Chapter 1 :Complex numbers and the Complex plane

الفصل الاول : الاعداد العقدية والمستوي العقدي

1- Complex numbers and Their properties

Definition: A complex number is any number of the form $z = a + ib$, where $i^2 = -1$ and a, b are real numbers.

Remarks:

1. The real number a in $z = a + ib$ is called the **real part** of z الجزء الحقيقيand denoted by $Re(z) = a$.
2. The real number b in $z = a + ib$ is called the **imaginary part** of z الجزء التخيليand denoted by $Im(z) = b$.
3. $i = \sqrt{-1}$, is called the **imaginary unit** الوحدة التخيلية.
4. The set of complex numbers is denoted by the symbol $\mathbb{C}$.
5. Because any real number a can be written as $z = a + 0$i, we see that the set $\mathbb{R}$ of real numbers is a subset of $\mathbb{C}$.
6. If z is complex number such that $Re(z) = 0$ and $Im(z) \neq 0$, then z is called pure imaginary number عدد تخيلي خالص.
7. $i^2 = -1, \quad i^3 = -i, \quad i^4 = 1$
   $i^{4n} = 1, \quad i^{4n+1} = i\ , \quad i^{4n+2} = -1\ , \quad i^{4n+3} = -i\ .$

Examples:

1. $z = 2 - 9i$
   $Re(z) = 2, \ Im(z) = -9$
2. $z = 7$
   $Re(z) = 7, \ Im(z) = 0$
3. $z = i$
   $Re(z) = 0, \ Im(z) = 1$
4. $z = 0$
   $Re(z) = 0, \ Im(z) = 0$

Definition: Complex numbers $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$ are **equal** if $a_1 = a_2$ and $b_1 = b_2$.

Fundamental operations on $\mathbb{C}$: العمليات الاساسية على مجموعة الاعداد العقدية

يمكننا ان نقوم بعملية جمع, طرح, ضرب او قسمة للأعداد العقدية معرفة لأي عددين عقديين كما يأتي

If $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$, these operations are defined as follows:

Addition: $z_1 + z_2 = (a_1 + ib_1) + (a_2 + ib_2) = (a_1 + a_2) + i(b_1 + b_2)$.

Subtraction: $z_1 - z_2 = (a_1 + ib_1) - (a_2 + ib_2) = (a_1 - a_2) + i(b_1 - b_2)$.

Multiplication: $z_1 . z_2 = (a_1 + ib_1)(a_2 + ib_2)$

$$= a_1 a_2 - b_1 b_2 + i(b_1 a_2 + a_1 b_2).$$

Division: $\frac{z_1}{z_2} = \frac{a_1 + ib_1}{a_2 + ib_2}$, $a_2 \neq 0$ or $b_2 \neq 0$

$$= \frac{a_1 a_2 + b_1 b_2}{{a_2}^2 + {b_2}^2} + i\ \frac{b_1 a_2 - a_1 b_2}{{a_2}^2 + {b_2}^2}\ .$$

The **commutative**, **associative**, and **distributive** laws hold for complex numbers:

Commutative laws: $z_1 + z_2 = z_2 + z_1$

$$z_1 z_2 = z_2 z_1$$

Associative laws: $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$

$$z_1(z_2 z_3) = (z_1 z_2) z_3$$

Distributive law: $z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$

Remarks:

1. The **zero** in the complex number system is the number $o + oi$ is an identity element for $+$, since for any complex number $z = a + ib$, we have $z + 0 = z$ اثبت ذلك باستخدام تعريف الجمع في نظام الاعداد العقدية
2. $\forall z = a + ib, \exists$ a unique additive invers which is $-z = -a - ib$ such that $z - z = 0$.
3. $1 = 1 + 0i$ is an identity element for $\cdot$, since for any complex number $z = a + ib$, we have $z.1 = z$ اثبت ذلك
4. Let $z = a + ib, a \neq 0$ or $b \neq 0$, z has multiplicative inverse z^{-1} $s.t$ $z^{-1} = \frac{1}{z} = \frac{a}{a^2 + b^2} + i\left(\frac{-b}{a^2 + b^2}\right)$.

5. $\forall z \neq 0, \exists z^{-1}\ s.t \quad zz^{-1} = 1$
6. $(\mathbb{C}, +, \cdot)$ is a field is called the field of complex numbers.

Definition: The **conjugate** مرافق العدد العقدي of a complex number $z = a + ib$, is the complex number $z = a - ib$ and is denoted by $\bar{z}$.

For example, if $z = 6 + \frac{\pi}{4}\, i$, then $\bar{z} = 6 - \frac{\pi}{4} i$, if $z = -7 - i$, then $\bar{z} = -7 + i$. If z is a real number, say, $z = 12$, then $\bar{z} = 12$.

Remark:

1. The definitions of addition and multiplication show that the sum and product of a complex number z with its conjugate $\bar{z}$ is a real number:

$$z + \bar{z} = (a + ib) + (a - ib) = 2a$$

$$z\bar{z} = (a + ib)(a - ib) = a^2 - i^2 b^2 = a^2 + b^2.$$

2. The difference of a complex number z with its conjugate $\bar{z}$ is a pure imaginary number: $z - \bar{z} = (a + ib) - (a - ib) = 2ib$.
3. ***Division:*** $\frac{z_1}{z_2} = \frac{\mathbf{z_1}}{\mathbf{z_2}} \cdot \frac{\overline{\mathbf{z_2}}}{\overline{\mathbf{z_2}}} = \frac{\mathbf{z_1}\overline{\mathbf{z_2}}}{\mathbf{z_2}\overline{\mathbf{z_2}}}$

properties:

i. $\bar{\bar{z}} = z$
ii. $\overline{z_1 \mp z_2} = \overline{z_1} \mp \overline{z_2}$
iii. $\overline{z_1 \cdot z_2} = \overline{z_1} \cdot \overline{z_2}$
iv. $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$
v. $Re(z) = \frac{z+\bar{z}}{2}, \quad Im(z) = \frac{z-\bar{z}}{2i}$

Examples: الأمثلة تحل اثناء المحاضرة

1. Find the value of $\frac{(1+i)(-1+2i)+(2-i)}{2-3i} - 2i$?
2. Solve the following equation $z^2 + z + 1 = 0$?
3. If $z_1 \cdot z_2 = 0$, then $z_1 = 0$ or $z_2 = 0$?
4. Find the value of $\frac{3i^{20} - i^{19}}{2i-1}$?

Remarks: Comparison with Real Analysis

Many of the properties of the real number system $\mathbb{R}$ hold in the complex number system $\mathbb{C}$, but there are some differences.

For example, we cannot compare two complex number $z_1 = a_1 + ib_1, b_1 \neq 0$, and $z_2 = a_2 + ib_2, b_2 \neq 0$, that is $z_1 < z_2$ or $z_2 \geq z_1$ has no meaning in $\mathbb{C}$ except in the special case when the two numbers z_1 and z_2 are real.

Exercises:

1. Let $z = 1 + 2i$ and $w = 2 - i$. Compute the following:
   - $z + 3w$
   - $Re(w^2 + w)$
   - $\overline{w} - z$
   - $z^2 + \bar{z} + i$
2. Find the real and imaginary parts of each of the following:
   - $\frac{z-a}{z+a}$ for any $a \in \mathbb{R}$
   - $\frac{3+5i}{7i+1}$
   - $\left(\frac{-1+i\sqrt{3}}{2}\right)^3$
   - i^n for any $n \in \mathbb{Z}$
3. Show that:
   - $i(1 - \sqrt{3}i)(\sqrt{3} + i) = 2(1 + \sqrt{3}i)$
   - $5i/(2 + i) = 1 + 2i$
   - $(-1 + i)^7 = -8(1 + i)$
   - $(1 + \sqrt{3}i)^{-10} = 2^{-11}(-1 + \sqrt{3}i)$
4. Find the following complex numbers in the form $a + ib$
   - $(4 - 7i)(-2 + 3i)$
   - $(5 + 2i)/(1 + i)$
   - $(1 - i)^3$
   - $1/i$

2- Complex Plane

Definition: The complex plane $\mathbb{C}$ is the set of all ordered pairs (a, b) of real numbers, with addition and multiplication defined by:

$$(a, b) + (c, d) = (a + c, b + d) \quad \text{and} \quad (a, b)(c, d) = (ac - bd, bc + ad)$$

Remark: A complex number $z = x + iy$ is uniquely determined by an ordered pair of real numbers (x, y).

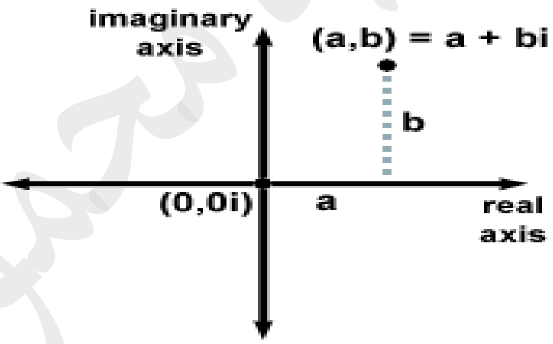
For example, $z = 3 - 9i = (3, -9)$,

$z = 8 = (8,0)$,

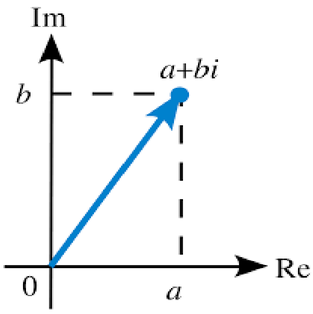
$z = i = (0,1)$,

$z = -7i = (0, -7)$.

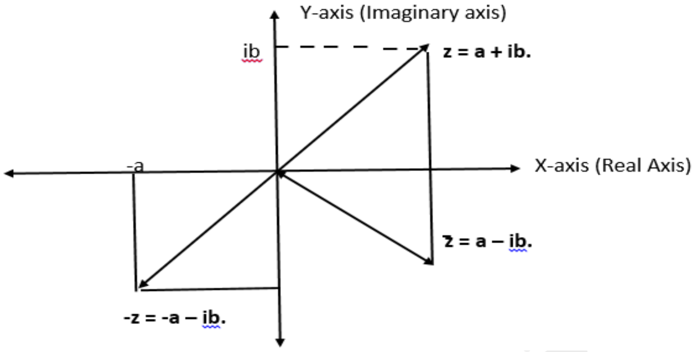
Remark: The coordinate plane is called the **complex plane** or simply the **z-plane**. The horizontal or $x - axis$ is called the real $axis$ because each point on that $axis$ represents a real number. The vertical or $y - axis$ is called the imaginary $axis$ because a point on that $axis$ represents a pure imaginary number.



Definition: The **modulus** of a complex number $z = a + ib$, is the real number $|z| = \sqrt{a^2 + b^2}$. The modulus $|z|$ of a complex number z is also called the **absolute value** of z.

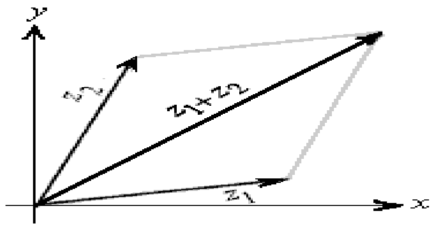


القيمة المطلقة للعدد العقدي هي المسافة بين النقطة التي تمثل العدد العقدي
القيمة المطلقة للعدد العقدي هي عدد حقيقي .ونقطة الاصل.



Properties:

1. $z\cdot\bar{z}=|z|^2$
2. $|\bar{z}|=|z|$
3. $|z|^2=[Re(z)]^2+[Im(z)]^2$
4. $|z|\geq|Re(z)|\geq Re(z)$ and $|z|\geq|Im(z)|\geq Im(z)$
5. $|z_1\cdot z_2|=|z_1|\cdot|z_2|$
6. $\left|\frac{z_1}{z_2}\right|=\frac{|z_1|}{|z_2|}$, $z_2\neq 0$
7. $|z_1+z_2|\leq|z_1|+|z_2|$, triangle inequality
8. $|z_1+z_2+\cdots+z_n|\leq|z_1|+|z_2|+\cdots+|z_n|$ تبرهن بالاستقراء الرياضي
   $i.e.,\left|\sum_{k=1}^{n}z_k\right|\leq\sum_{k=1}^{n}|z_k|$
9. $|z_1-z_2|\geq|z_1|-|z_2|$
10. $\left||z_1|-|z_2|\right|\leq|z_1+z_2|$



Proof(7):

$$|z_1+z_2|^2=(z_1+z_1)\cdot\overline{(z_1+z_2)}\quad=(z_1+z_2)\cdot(\bar{z_1}+\bar{z_2})$$

$$=z_1\bar{z_1}+z_1\bar{z_2}+z_2\bar{z_1}+z_2\bar{z_2}$$

$$=|z_1|^2+z_1\bar{z_2}+\overline{\bar{z_2}z_1}+|z_2|^2=|z_1|^2+|z_2|^2+2Re(z_1\bar{z_2})$$

$$\leq|z_1|^2+|z_2|^2+2|z_1\bar{z_2}|=|z_1|^2+|z_2|^2+2|z_1|\cdot|\bar{z_2}|$$

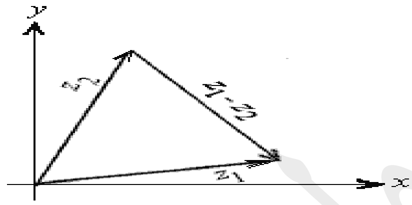
$$=|z_1|^2+|z_2|^2+2|z_1|\cdot|z_2|=(|z_1|+|z_2|)^2$$

Therefore $|z_1 + z_2| \leq |z_1| + |z_2|$

Remark: $|z_1 - z_2|$ is the distance in the plane on the complex number z_1 from the complex number z_2.

If $z_1 = x_1 + iy_1, \quad z_2 = x_2 + iy_2$

$$|z_1 - z_2| = |(x_1 - x_2) + i(y_1 - y_2)| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$



Proof(9): $|z_1| = |z_1 - z_2 + z_2| \leq |z_1 - z_2| + |z_2|$

$$\rightarrow |z_1| - |z_2| \leq |z_1 - z_2|$$

Proof (10): $|z_1| = |z_1 + z_2 - z_2| \leq |z_1 + z_2| + |-z_2| = |z_1 + z_2| + |z_2|$

$$|z_1| - |z_2| \leq |z_1 + z_2| \quad \text{———} (1)$$

$$|z_2| = |z_2 + z_1 - z_1| \leq |z_1 + z_2| + |-z_1| = |z_1 + z_2| + |z_1|$$

$$|z_2| - |z_1| \leq |z_1 + z_2|$$

$$|z_1| - |z_2| \geq -|z_1 + z_2| \quad \text{———} (2)$$

From (1) and (2) we get $-|z_1 + z_2| \leq |z_1| - |z_2| \leq |z_1 + z_2|$

Therefore $\big||z_1| - |z_2|\big| \leq |z_1 + z_2|$.

Remark: $z_1 < z_2$ in general has no meaning but $|z_1| < |z_2|$ means the point corresponding to z_1 is closer to the origin than the point corresponding to z_2.

Notation: Let $z_0 \in \mathbb{C}, r > 0$

$\boldsymbol{S_r(z_0)} = \{z \in \mathbb{C}: |z - z_0| = r\}$ is the **circle** دائرة with center z_0 and radius r.

$\boldsymbol{B_r(z_0)} = \{z \in \mathbb{C}: |z - z_0| < r\}$ is the **ball** كرة with center z_0 and radius r.

$\boldsymbol{D_r(z_0)} = \{z \in \mathbb{C}: |z_1 - z_0| \leq r\}$ is the **disk** قرص with center z_0 and radius r.

Example: The equation $|z-1+3i|=2$ represents the circle whose center is $z_0=(1,-3)$ and whose radius is $r=2$.

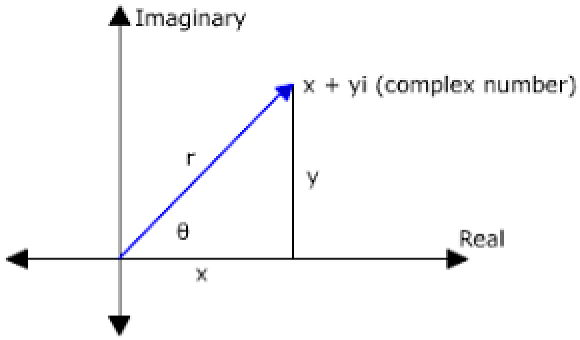
Exercises:

1. Find the modulus (absolute value) and conjugate of each of the following:
   - $(1-i)^2$
   - $i(2-i)-4\left(1+\frac{1}{4}i\right)$
   - $\frac{2i}{3-4i}$
   - $\frac{1-2i}{1+i}+\frac{2-i}{1-i}$
   - $\frac{(4+3i)(1+i)}{7-i}$
2. Show that:
   - $\overline{z+3i}=z-3i$.
   - $\overline{iz}=-i\bar{z}$
   - $\overline{(2+i)^2}=3-4i$
   - $\left|(2\bar{z}+5)(\sqrt{2}-i)\right|=\sqrt{3}|2z+5|$
3. Show that
   a) $\overline{z_1z_2z_3}=\bar{z}_1\bar{z}_2\bar{z}_3$
   b) $\overline{z^4}=\bar{z}^4$
4. Find the complex number E which is satisfies the following:
   $|E|=1$ and $Re(E^2)=0$.
5. Show that equation $|z-z_0|=R$ of circle, centered at z_0 with radius R, can be written as: $|z|^2-2Re(z\overline{z_0})+|z_0|^2=R^2$.
6. z is real number if and only if $z=\bar{z}$.
7. z is either real or pure imaginary if and only if $(\bar{z})^2=z^2$.

3- Polar form of complex numbers

Let r and θ be polar coordinates of the point (x, y) that corresponds to a nonzero complex number $z = x + iy$. Since $x = r\cos\theta$ and $y = r\sin\theta$, the number z can be written in polar form as:

$$z = r(\cos\theta + i\sin\theta).$$



Remarks:

1. If $z = 0$, then θ is not defined and hence we cannot write $z = 0$ in polar form.
2. In complex analysis we assume $r \geq 0$ and $r = |z| = \sqrt{x^2 + y^2}$.
3. The real number θ represents the angle, measured in radians, that z makes with positive real axis.
4. θ is called an argument of z, which has infinite number of values, both positive and negative, that differ by integer multiple of 2π.
5. The values of θ can be determined by specifying the quadrant containing $z = x + iy$ and by using one of the following three equations: $\tan\theta = \frac{y}{x}$, $\cos\theta = \frac{x}{|z|}$, $\sin\theta = \frac{y}{|z|}$.
6. The set of all arguments of z is denoted by $\arg(z)$.
7. The principal value of $\arg(z)$ is the unique θ, such that $-\pi < \theta \leq \pi$ and it is denoted by $\mathrm{Arg}(z)$. $\arg(z) = \mathrm{Arg}(z) + 2n\pi$, $n \in \mathbb{Z}$.
8. $$\mathrm{Arg}(z) = \begin{cases} \tan^{-1}(y/x) & x > 0 \\ \pi + \tan^{-1}(y/x) & x < 0,\ y \geq 0 \\ -\pi + \tan^{-1}(y/x) & x < 0,\ y < 0 \\ \pi/2 & x = 0,\ y > 0 \\ -\pi/2 & x = 0,\ y < 0 \end{cases}$$

Example (1) Find $\operatorname{Arg}(5)$, $\operatorname{Arg}(-5)$, $\operatorname{Arg}(2i)$, $\operatorname{Arg}(-3i)$?

Solution : $\operatorname{Arg}(5) = \tan^{-1}(y/x) = \tan^{-1}(0) = 0.$

$\operatorname{Arg}(-5) = \pi + \tan^{-1}(0) = \pi.$

$\operatorname{Arg}(2i) = \pi/2.$

$\operatorname{Arg}(-3i) = -\pi/2.$

Example (2) Find the principal argument of $z = 1 + i$?

Solution : $\operatorname{Arg}(z) = \tan^{-1}(y/x) = \tan^{-1}(1) = \frac{\pi}{4}$

Example (3) If $z = \frac{1+i\sqrt{3}}{1-i\sqrt{3}}$, find $\arg(z)$?

Solution : $z = \frac{1+i\sqrt{3}}{1-i\sqrt{3}} = \frac{-1+\sqrt{3}i}{2}$

$$\operatorname{Arg}(z) = \pi + \tan^{-1}(y/x) = \pi + \tan^{-1}\left(\frac{\sqrt{3}/2}{-1/2}\right) = \pi + \tan^{-1}(-\sqrt{3})$$

$$= \pi - \tan^{-1}(\sqrt{3}) = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$$

$\arg(z) = \frac{2\pi}{3} + 2n\pi, \quad n \in \mathbb{Z}.$

Example (4) Express $-\sqrt{3} - i$ in polar form.

Solution : يحل اثناء المحاضرة

Example (5) Write the complex number $z = 2 - 2i$ in polar form.

Solution : يحل اثناء المحاضرة

Proposition : Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$, $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

Then, $z_1 z_2 = r_1 r_2(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2))$ and

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}(\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2))$$

Proof : $z_1 \cdot z_2 = r_1(\cos\theta_1 + i\sin\theta_1) \cdot r_2(\cos\theta_2 + i\sin\theta_2)$

$$= r_1 r_2[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)]$$

$$= r_1 r_2(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2))$$

And

$$\frac{z_1}{z_2} = \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)} \cdot \frac{(\cos\theta_2 - i\sin\theta_2)}{(\cos\theta_2 - i\sin\theta_2)}$$

$$= \frac{r_1[(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2)]}{r_2[(\cos\theta_2)^2 + (\sin\theta_2)^2]}$$

$$= \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)] \qquad \blacksquare$$

Remark : 1) $arg(z_1 \cdot z_2) = arg(z_1) + arg(z_2)$

2) $arg\left(\frac{z_1}{z_2}\right) = arg(z_1) - arg(z_2)$

Proposition : Let $z_j = r_j(\cos\theta_j + i\sin\theta_j), \quad j = 1,2,3,\dots,n$

Then $z_1.z_2 \cdots z_n = r_1 \cdot r_2 \cdots r_n(\cos(\theta_1 + \theta_2 + \cdots \theta_n) + i\sin(\theta_1 + \theta_2 + \cdots \theta_n))$

If $z_1 = z_2 = \cdots = z_n = z = r(cos\,\theta + i\sin\theta)$, then

$$z^n = r^n(\cos(n\theta) + i\sin(n\theta)) \qquad \cdots \quad (1)$$

De Moivere's Formula: When $z = (cos\,\theta + i\sin\theta)$, we have $|z| = r = 1$,

And so (1) yields $\quad (cos\,\theta + i\sin\theta)^n = (\cos(n\theta) + i\sin(n\theta)) \quad \cdots \quad (2)$

This last result is known as **de Moiver's formula**.

Example : From (2), with $\theta = \frac{\pi}{6}$, $\cos\theta = \sqrt{3}/2$ and $\sin\theta = 1/2$:

$$\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)^3 = \cos 3\theta + i\sin 3\theta = \cos\left(3 \cdot \frac{\pi}{6}\right) + i\sin\left(3 \cdot \frac{\pi}{6}\right)$$

$$= \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = i.$$

Example : Let $z = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$, find z^2?

Solution: $z^2 = \left[2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)\right]^2 = 2^2\left(\cos 2.\frac{\pi}{6} + i\sin 2.\frac{\pi}{6}\right)$

$$= 4\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) = 4\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = 2 + 2\sqrt{3}\,i\,.$$

Euler's Formula : By Taylor's Series $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

If we take $x = i\theta$, then $e^{i\theta} = \cos\theta + i\sin\theta$

Let $z = r(\cos\theta + i\sin\theta) = re^{i\theta}$ $\cdots$ $(Euler\ formula)$

Remark : Let $z_1 = r_1 e^{i\theta_1}$, $z_2 = r_2 e^{i\theta_2}$. Then,

1. $z_1 z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)}$
2. $\frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)}$
3. If $z = re^{i\theta}$, then $z^n = r^n e^{in\theta}$ $n \in \mathbb{Z}^+$
4. The invers of any non-zero complex number $z = re^{i\theta}$ is:
   $z^{-1} = \frac{1}{z} = \frac{1}{r} e^{-i\theta}$
5. $z^0 = \left(re^{i\theta}\right)^0 = r^0 e^{i0\theta} = 1$
   If $n = -1, -2, \ldots$ then, $z^n = r^n e^{in\theta}$ (why?).

Exercises:

1. Write in polar form :
   a. $2i$ $b.$ $1+i$ $c.$ $-3+\sqrt{3}\,i$ $d.$ $-i$
2. Write in rectangular form :
   a. $\sqrt{2}\,e^{i\frac{3\pi}{4}}$ $b.$ $34\,e^{i\frac{\pi}{2}}$ $c.$ $-\ e^{i\,250\pi}$ $d.$ $2\,e^{i\,4\pi}$.
3. (a) Find a polar form of $(1+i)\left(1+i\sqrt{3}\right)$
   (b) Use the result of (a) to find $\cos\left(\frac{7\pi}{12}\right)$ and $\sin\left(\frac{7\pi}{12}\right)$
4. Use Euler's formula to show the following:
   - $\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$
   - $\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$
   - $sin^3\theta = \frac{3}{4}\sin\theta - \frac{1}{4}\sin 3\theta$
   - $cos^4\theta = \frac{1}{8}\cos 4\theta + \frac{1}{2}\cos 2\theta + \frac{3}{8}$
5. Use de Moiver's formula to derive:
   - $\cos 2\theta = cos^2\theta - sin^2\theta$
   - $\sin 2\theta = 2\sin\theta\cos\theta$
   - $\cos 3\theta = cos^3\theta - 3\cos\theta\, sin^2\theta$
   - $\sin 3\theta = 3cos^2\theta\sin\theta - sin^3\theta$

6. Find $\arg(1 - i\sqrt{3})^3$?
7. Show that $\left|e^{i\theta}\right| = 1$
8. Find the rectangular form of $\left(\sqrt{3} + i\right)^5$
9. Find the rectangular form of $(-1 + i)^{100}$
10. Find the rectangular form of $\left[2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)\right]^2$

4- Powers and Roots of complex numbers القوى والجذور للأعداد العقدية

Definition: Let $z_0 = r_0 e^{i\theta_0}$, non-zero complex number $(z_0)^{\frac{1}{n}}$, n is positive integer is called (**nth- root**) for equation $z^n = z_0$

$$z^n = z_0$$

$$\left(re^{i\theta}\right)^n = z_0 = r_0 e^{i\theta_0} = r_0 e^{i(\theta_0 + 2k\pi)}, k \in \mathbb{Z}$$

$$r^n e^{in\theta} = r_0 e^{i(\theta_0 + 2k\pi)}, k \in \mathbb{Z}$$

$$r^n = r_0 , \qquad n\theta = \theta_0 + 2k\pi$$

$$r = \sqrt[n]{r_0} , \ \theta = \frac{\theta_0 + 2k\pi}{n}$$

Therefor, $(z_0)^{\frac{1}{n}} = \sqrt[n]{r_0}\, e^{i\left(\frac{\theta_0}{n} + \frac{2k\pi}{n}\right)}, k \in \mathbb{Z}$

The distinct roots obtained when $k = 0,1,2,\dots,n-1$.

The distinct roots denoted by c_k

$$c_k = \sqrt[n]{r_0}\, e^{i\left(\frac{\theta_0}{n} + \frac{2k\pi}{n}\right)}, \quad k = 0,1,2,\dots,n-1$$

$$= \sqrt[n]{r_0}\left[\cos\left(\frac{\theta_0}{n} + \frac{2k\pi}{n}\right) + i\sin\left(\frac{\theta_0}{n} + \frac{2k\pi}{n}\right)\right]$$

Remark: If $n = 2$, then the two roots lie at the opposite ends $c_1 = -c_0$.

Examples: تحل اثناء المحاضرة

1) Find the square roots of $\sqrt{3} + i$?
2) Find the 2-nd roots of $2i$?
3) Find the cube roots of $-8i$?
4) Find the square roots of $1 - \sqrt{3}i$?
5) Find the 3-rd roots of 1 ?

The nth roots of unity

The nth roots of 1 are given by :

$$c_k = e^{\frac{2k\pi}{n}i}, \quad k = 0, 1, 2, \dots, n-1.$$
$$c_0 = 1.$$

Let $w_n = c_1 = e^{\frac{2\pi}{n}i}$. Then

$$c_2 = e^{\frac{4\pi}{n}i} = w_n{}^2, \dots, c_{n-1} = e^{\frac{2(n-1)\pi}{n}i} = w_n{}^{n-1}$$

Therefore, the nth roots of unity are: $1, w_n, w_n{}^2, \dots, w_n{}^{n-1}$, where $w_n = e^{\frac{2\pi}{n}i}$.

The nth roots of z_0

The nth roots of $z_0 = r_0 e^{i\theta_0}$ are given by :

$$c_k = \sqrt[n]{r_0}\, e^{i\left(\frac{\theta_0}{n}+\frac{2k\pi}{n}\right)}, \quad k = 0,1,2, \dots, n-1$$

They can be written as :

$$c_k = \sqrt[n]{r_0}\ e^{i\frac{\theta_0}{n}} \cdot e^{\frac{2k\pi}{n}i} = c_0\, w_n{}^k, \quad k = 0,1,2, \dots, n-1$$

Where $c_0 = \sqrt[n]{r_0}\ e^{i\frac{\theta_0}{n}}$ is the principal root of z_0 and $w_n = e^{\frac{2\pi}{n}i}$.

Examples: تحل اثناء المحاضرة

1) Find the square roots of i ?
2) Find all values of $\left(2 + 2\sqrt{3}i\right)^{\frac{1}{4}}$?
3) Find all values of $\left(-4\sqrt{3} + 4i\right)^{\frac{1}{3}}$?

Remark: If $n < 0$, then $(z_0)^{\frac{1}{n}} = \left(\frac{1}{z_0}\right)^{-\frac{1}{n}}$.

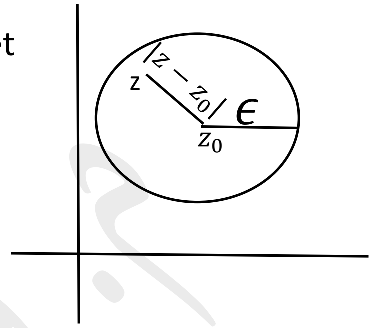
Exercises:

1. Find all solution to the following equation:
   b. $z^6 = 1$
   c. $z^4 = -16$
   d. $z^4 = -8 - 8\sqrt{3}\, i$
   e. $z^6 = 8$
   f. $z^5 = -39$
2. Find the cube roots of $-1 + i$?
3. Find the two values of $(7 + 24i)^{1/2}$?
4. Find all solutions of the equation $z^4 + 1 = 0$?
5. Show that the two roots of $(1 + i)^{\frac{1}{2}}$ are $\pm a$,
   Where, $a = \sqrt[4]{2}\left[\cos\frac{\pi}{8} + i\sin\frac{\pi}{8}\right]$
6. If $w \neq 1$ is the nth root of 1 then, show that
   $1 + w + w^2 + \cdots + w^{n-1} = 0$

5- Sets of Points in the Complex Plane مجموعات النقاط في المستوي العقدي

(المناطق في المستوي العقدي) لإنهاء الفصل الأول والذي يعتبر مقدمة عامة لمفهوم الاعداد العقدية نحتاج ان نعرف المنطقة والجوار للنقطة (نقصد بالجوار مجموعة النقاط القريبة من النقطة) لكي نكون مستعدين لدراسة الدالة العقدية في الفصل القادم والغايات عند نقطة.

<u>**Definition:**</u> An ϵ −**neighborhood** جوار of a points z_0 is the set



$$N_\epsilon(z_0) = \{z \in \mathbb{C} : \ |z - z_0| < \epsilon\}$$

Consisting of all points lying inside the circle $|z - z_0| < \epsilon$.

<u>**Definition:**</u> A **deleted ϵ −neighborhood** of a points z_0 an ϵ −neighborhood of a points z_0 except z_0 itself. That is it is the set $N_\epsilon(z_0) \setminus \{z_0\}$

i.e. $$N_\epsilon(z_0) = \{z \in \mathbb{C} : \ 0 < |z - z_0| < \epsilon\}$$

Consisting of all points lying inside the circle $|z - z_0| < \epsilon$.

<u>**Definition:**</u> A points z_0 is said to be an **interior point** نقطة داخلية of a set S of complex numbers if there is an ϵ −neighborhood $N_\epsilon(z_0)$ of z_0 such that $N_\epsilon(z_0) \subseteq S$.

<u>**Definition:**</u> A points z_0 is said to be an **exterior point** نقطة خارجية of a set S of complex numbers if there is an ϵ −neighborhood $N_\epsilon(z_0)$ of z_0 such that $N_\epsilon(z_0) \cap S = \emptyset$.

<u>**Definition:**</u> A points z_0 is said to be a **boundary point** نقطة حدودية of a set S of complex numbers if it is neither an interior nor an exterior point.

<u>**Definition:**</u> The set of all boundary points of a set S is called the **boundary of S**.

<u>**Example:**</u> Let $S = \{z : |z| < 1\}$.

Then the <u>boundary of S</u> is the set: $S' = \{z : |z| = 1\}$.

$z_1 = \frac{1+i}{2}$ is an <u>interior point</u> of S.

$z_2 = 1 + i$ is an <u>exterior point</u> of S.

Definition: Let S be a set of complex numbers

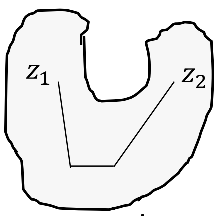
1. If every point z of a set S is an interior point, then S is said to be an **open set**.
2. A subset S of $\mathbb{C}$ is said to be **closed set** if S complement $(\mathbb{C} - S)$ is open.
3. The **closure** of S , denoted $(\bar{S})$, is the closed set consisting of all points in S together with the boundary of S.

Examples:

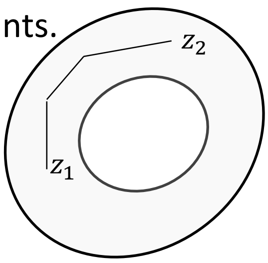
(1) $\mathbb{C}, \emptyset$, the only sets that are both open and closed.
(2) $S = \{z: Im(z) < 1\}$ is open set. The closure of S is the set $\bar{S} = \{z: m(z) \leq 1\}$.
(3) $B_r(z_0)$ is open set but not closed.
(4) $D_r(z_0)$ is closed set but not open.
(5) $S_r(z_0)$ is closed set but not open.
(6) $B_1(0) \cup \{1\}$ is not open , not closed.

Definition:

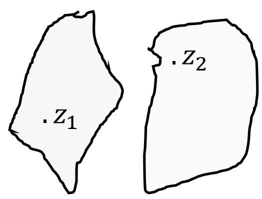
1. If any pair of points z_1 and z_2 in a set S can be connected by a polygonal line that consists of a finite number of line segments joined end to end that lies entirely in the set, then the set S is said to be **connected**.
2. An open connected set is called a **domain**.
3. A **region** is a set of points in the complex plane with all, some, or none of its boundary points.



Connected



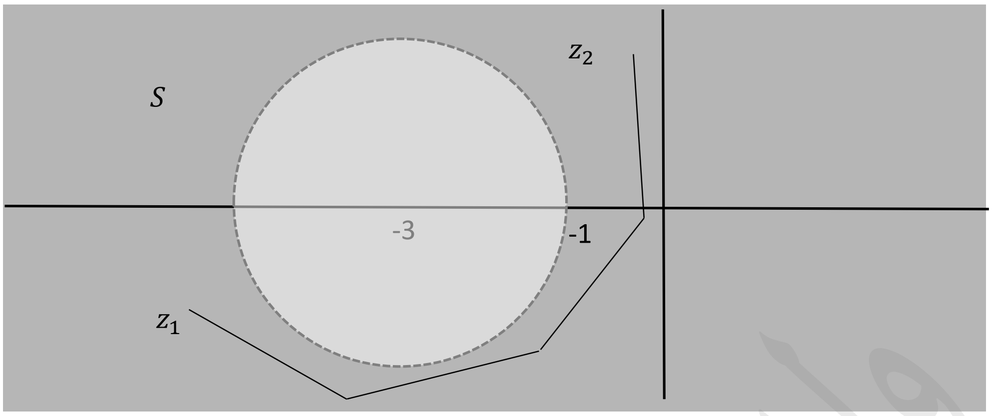
Connected



Disconnected

Example: Sketch $S = \{z: |z+3| > 2\}$ and determine whether it is a domain or not.

Sol: S is open and connected $\Longrightarrow$ S is domain $\Longrightarrow$ S is region.



Definition:

1. A set S is said to be **bounded** if there is a real number $M > 0$ such that $|z| < M$ for all $z \in S$.
2. A point z_0 is called an **accumulation point** of a set S if each deleted neighborhood of z_0 contains at least one point of S.

Example: Let $S = \left\{\frac{i^n}{n}\ ,\ n = 1, 2, \dots\right\}$, then 0 is the only accumulation point.

Exercises:

1. Sketch the following sets in the complex plane and determine whether they are open, closed, a domain, bounded or connected :
   g. $S = \{z: |z-i| > 1\}$
   h. $S = \{z: 1 \leq |z-1-i| < 2\}$
   i. $S = \{z: Re(z) < -1\}$
   j. $S = \{z: Im(z) > 3\}$
   k. $S = \{z: 2 < Re(z-1) < 4\}$
2. Determine the accumulation points of each of the following sets:
   - $z_n = i^n \ \ (n = 1, 2, 3, \dots)$
   - $z_n = (-1)^n (1+i)\frac{n-1}{n} \ \ (n = 1, 2, 3, \dots)$

Chapter 2: ANALYTIC FUNCTIONS

1- FUNCTIONS OF A COMPLEX VARIABLE

Definition:

Let S be a set of complex numbers. A **function** f defined on S is a rule that assigns to each z in S a complex number w. The number w is called the value of f at z and is denoted by $f\ (z)$, that is, $w\ =\ f\ (z)$.

Remark:

1. If f is a function defined on S, then S is the domain of f.
2. If the domain is not given explicitly, then we agree that it's the largest possible set.

Example: Let $f(z) = \frac{1}{z-1}$.

Then f is a function of complex variable with domain $S = \mathbb{C} - \{1\}$.

Example: Let $f(z) = \frac{1}{z^2+1}$.

Then f is a function of complex variable with domain $S = \mathbb{C} - \{i, -i\}$.

Real and imaginary parts of a function

Let $w\ =\ u\ +\ i\ v$ is the value of a function f at $z\ =\ x\ +\ iy$, so that

$$u + \ iv = \ f\ (x + iy).$$

Both of the real variables u and v depends on x and y. Thus

$$f(z) = u(x, y) + iv(x, y)$$

Where u and v are real valued functions of two real variables x and y.

Similarly, if $z = re^{i\theta}$, then $f(z) = u(r, \theta) + iv(r, \theta)$

Example: Write $f(z) = z^2 + 2z$ in the forms: $f(z) = u(x, y) + iv(x, y)$ and $f(z) = u(r, \theta) + iv(r, \theta)$.

Solution:

- $z = x + iy$

$$f(z) = z^2 + 2z = (x + iy)^2 + 2(x + iy)$$
$$= x^2 - y^2 + 2xyi + 2x + 2yi$$
$$= (x^2 - y^2 + 2x) + (2xy + 2y)i$$
$$= u(x, y) + iv(x, y)$$

- $z = re^{i\theta}$

$$f(z) = z^2 + 2z = r^2e^{2\theta i} + 2re^{i\theta}$$
$$= r^2(\cos 2\theta + i \sin 2\theta) + 2r(\cos\theta + i\sin\theta)$$
$$= r^2\cos 2\theta + 2r\cos\theta + i(r^2\sin 2\theta + 2r\sin\theta)$$
$$= u(r, \theta) + iv(r, \theta)$$

Example: If $v(x, y) = 0$ for all z, then $f(z) = u(x, y)$ is called a real valued function of a complex variable.

Example: $f(z) = |z|^2 = x^2 + y^2$ is a real valued function of a complex variable.

Polynomials and Rational Function:

1. If n is a nonnegative integer, and if $a_0, a_1, \cdots, a_n$ are complex numbers, such that $a_n \neq 0$, then the function
   $f(z) = a_0 + a_1 z + \cdots + a_n z^n$
   Is a polynomial with degree n.
   The domain of a polynomial function is the entire z-plane.
2. If $P(z)$ and $Q(z)$ are polynomials, then $f(z) = \frac{P(z)}{Q(z)}$
   is rational function.The domain of a rational function is the set
   $\{z \in \mathbb{C} : Q(z) \neq 0\}$

Example: $f(z) = iz^4 - (1+i)z^3 + 2z - 5 - 3i$
Is a polynomial of degree 4 whose domain is $\mathbb{C}$.

Example: $f(z) = \frac{2z^2+3iz+4}{z^3-(1+i)z^2+iz}$ Is a rational function whose domain is $\mathbb{C} - \{0,1,i\}$.

Since $z^3 - (1+i)z^2 + iz = 0$

$z(z^2 - (1+i)z + i) = 0$, $z = 0$ or $z^2 - (1+i)z + i = 0$

$(z - i)(z - 1) = 0 \implies z = i$ or $z = 1$.

Example: $f(z) = x^2 + y^2 + 2i(x + y)$.

Write $f(z)$ in terms of z.

Solution:

$x = \frac{z+\bar{z}}{2}$, $y = \frac{z-\bar{z}}{2i}$

$$f(z) = \left(\frac{z+\bar{z}}{2}\right)^2 + \left(\frac{z-\bar{z}}{2i}\right)^2 + 2i\left(\frac{z+\bar{z}}{2} + \frac{z-\bar{z}}{2i}\right)$$

$$= \frac{1}{4}(z^2 + 2z\bar{z} + \bar{z}^2) - \frac{1}{4}(z^2 - 2z\bar{z} + \bar{z}^2) + i(z+\bar{z}) + z - \bar{z}$$

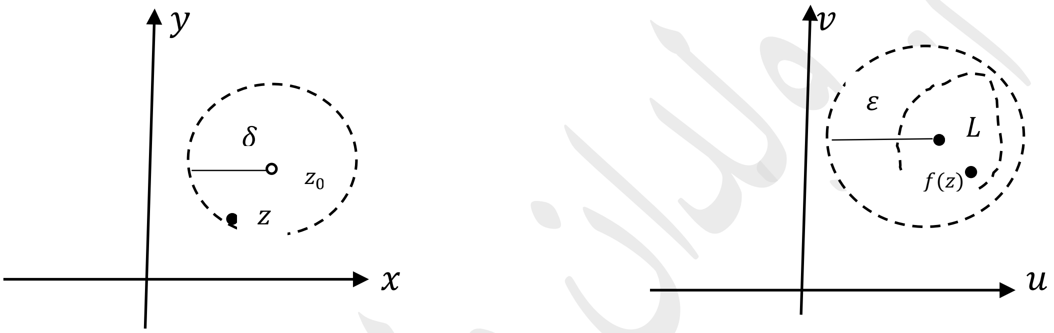
$$= z\bar{z} + z - \bar{z} + i(z + \bar{z})$$

Thus $f(z) = |z|^2 + (1+i)z - (1-i)\bar{z}$.

2- LIMITS OF A COMPLEX FUNCTION

Definition: (Limit of a Complex Function)

Suppose that a complex function f is defined in a deleted neighborhood of z_0 and suppose that L is a complex number. The **limit** of f as z tends to z_0 exists and is equal to L, written as $\lim_{z\to z_0} f(z) = L$, if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $|f(z) - L| < \varepsilon$ whenever $0 < |z - z_0| < \delta$.



z-plane w-plane

Examples:

تحل الأمثلة اثناء المحاضرة

1. Show that $\lim_{z\to z_0} z = z_0$?
2. Prove that $\lim_{z\to i}(3iz + 2) = -1$?
3. Prove that $\lim_{z\to 1+i}(2 + i)z = 1 + 3i$? H.W

Remarks:

1. The definition of limit requires f to be defined in some deleted neighborhood of a point z_0. Such a deleted neighborhood always exists when z_0 is an interior point of the domain of f.
2. We can extended the definition of limit to the case in which z_0 is a boundary point of the domain of f. in this case we require $|f(z) - L| < \varepsilon$, only for z lies in both the domain of f and $0 < |z - z_0| < \delta$.
3. If $\lim_{z\to z_0} f(z) = L$ exists, then it is unique.
4. The symbol $z \to z_0$ means that z is allowed to approach z_0 in any arbitrary manner. Thus if the limit approach different value from different directions, then the limit does not exist.

Exercise : Show that $\lim_{z\to 0} \frac{z}{\bar{z}}$ does not exist?

Theorem 1: Suppose that $f(z) = u(x,y) + iv(x,y), z_0 = x_0 + iy_0$, and $L = A + iB$. Then $\lim_{z\to z_0} f(z) = L$ if and only if

$\lim_{(x,y)\to(x_0,y_0)} u(x,y) = A$ and $\lim_{(x,y)\to(x_0,y_0)} v(x,y) = B$.

Example:

Use theorem 1 to compute $\lim_{z\to 1+i} (z^2 + i)$.

Solution.

Since $f(z) = z^2 + i = x^2 - y^2 + (2xy + 1)i$

$u(x,y) = x^2 - y^2, v(x,y) = 2xy + 1$ and $z_0 = x_0 + iy_0 = 1 + i$.

$A = \lim_{(x,y)\to(1,1)} x^2 - y^2 = 1^2 - 1^2 = 0$

and

$B = \lim_{(x,y)\to(1,1)} 2xy + 1 = 2.1.1 + 1 = 3,$

and so $L = A + iB = 0 + i(3) = 3i$

Therefor $\lim_{z\to 1+i} (z^2 + i) = 3i$

Theorem 2: Assume that $\lim_{z\to z_0} f(z) = L$ and $\lim_{z\to z_0} g(z) = M$. Then

a) $\lim_{z\to z_0} cf(z) = cL$, c a complex constant,
b) $\lim_{z\to z_0} [f(z) \pm g(z)] = L \pm M$,
c) $\lim_{z\to z_0} [f(z)g(z)] = LM$,
d) $\lim_{z\to z_0} \left[\frac{f(z)}{g(z)}\right] = \frac{L}{M}$ provided $M \neq 0$.

Corollary :

1) If $P(z)$ is a polynomial, then $\lim_{z\to z_0} P(z) = P(z_0)$.
2) If $P(z)$ and $Q(z)$ are polynomials such that $Q(z_0) \neq 0$, then $\lim_{z\to z_0} \frac{P(z)}{Q(z)} = \frac{P(z_0)}{Q(z_0)}$.

Example : Compute the limits

a) $\lim_{z\to i} \frac{(3+i)z^4 - z^2 + 2z}{z+1}$
b) $\lim_{z\to 1+\sqrt{3}i} \frac{z^2 - 2z + 4}{z - 1 - \sqrt{3}i}$ H.W

Solution.

a) $\lim_{z\to i}\frac{(3+i)z^4-z^2+2z}{z+1}=\frac{\lim_{z\to i}\left((3+i)z^4-z^2+2z\right)}{\lim_{z\to i}(z+1)}$

$$=\frac{(3+i)\lim_{z\to i}z^4-\lim_{z\to i}z^2+2\lim_{z\to i}z}{\lim_{z\to i}(z+1)}$$

$$=\frac{(3+i)(1)-(-1)+2i}{i+1}$$

$$=\frac{4+3i}{1+i}=\frac{7}{2}-\frac{1}{2}i$$

3- CONTINUITY OF COMPLEX FUNCTIONS

Definition: A complex function f is **continuous** at a point z_0 if

$\lim_{z\to z_0} f(z)=f(z_0)$

A complex function f is **continuous** at a point z_0

if each of the following three conditions hold:

i. $\lim_{z\to z_0} f(z)$ exists,

ii. f is defined at z_0 , and

iii. $\lim_{z\to z_0} f(z)=f(z_0)$.

If a complex function f is not continuous at a point z_0 then we say that f is discontinuous at z_0. For example, the function $f(z)=\frac{1}{1+z^2}$ is discontinuous at $z=i$ and $z=-i$.

Example: Checking Continuity of the function $f(z)=z^2-iz+2$ at the point $z_0=1-i$.

Solution.

$\lim_{z\to z_0} f(z)=\lim_{z\to 1-i}(z^2-iz+2)=(1-i)^2-i(1-i)+2=1-3i.$

$f(z_0)=f(1-i)=(1-i)^2-i(1-i)+2=1-3i.$

Since $\lim_{z\to z_0} f(z)=f(z_0)$

Therefore

$f(z)=z^2-iz+2$ is continuous at the point $z_0=1-i$.

Definition: A complex function f is **continuous** in a region $S\subseteq\mathbb{C}$ if

f is continuous at each $z\in S$.

Example: A polynomial function $f(z) = a_0 + a_1 z + \cdots + a_n z^n$ is continuous in the region $\mathbb{C}$.

Proposition: Let f and g are continuous functions at z_0. Then

a) $f(z) \pm g(z)$ is continuous at z_0,
b) $f(z)g(z)$ is continuous at z_0,
c) $f(z)/g(z)$ is continuous at z_0, provided that $g(z_0) \neq 0$.

Example: The function $f(z) = \frac{z^2 - iz + 5i + 1}{z}$ is continuous in the region $\mathbb{C}\backslash\{0\}$.

Remark: A function $f(z) = u + iv$ is continuous at a point

$z_0 = x_0 + iy_0$ if and only if u and v are continuous at (x_0, y_0)

Example: The function

$f(z) = e^x \cos y + ie^x \sin y$

is continuous everywhere in the complex plane $\mathbb{C}$ since

$u(x, y) = e^x \cos y$, and $v(x, y) = e^x \sin y$

are continuous at each point (x, y) in the plane.

Exercises:

In problems 1-2, show that the function f is continuous at the given point.

1. $f(z) = \frac{Re(z)}{z + iz} - 2z^2;\ z_0 = e^{i\pi/4}$
2. $f(z) = \begin{cases} \frac{z^3 - 1}{z - 1}, & |z| \neq 1 \\ 3, & |z| = 1 \end{cases};\ z_0 = 1$

In problems 1-2, show that the function f is discontinuous at the given point.

1. $f(z) = \begin{cases} \frac{z^3 - 1}{z - 1}, & |z| \neq 1 \\ 3, & |z| = 1 \end{cases};\ z_0 = i$
2. $f(z) = \begin{cases} \frac{z}{|z|}, & z \neq 0 \\ 1, & z = 0 \end{cases};\ z_0 = 0.$

4- DIFFERENTIABILITY OF COMPLEX FUNCTIONS

Definition: (Derivative of Complex Function)

Suppose the complex function f is defined in a neighborhood of a point z_0. The **derivative** of f at z_0, denoted by $f'(z_0)$ or $\frac{d}{dz}f(z)|_{z_0}$, is defined by the equation

$$f'(z_0) = \lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0}$$
$$= \lim_{\triangle z\to 0}\frac{f(z_0+\triangle z)-f(z_0)}{\triangle z}$$

Provided this limit exists.

If $f'(z_0)$ exists, then f is said to be differentiable at z_0.

Example: Use definition of derivative to find $f'(z_0)$ if $f(z) = z^2 - 5z$.

Solution.

$$f'(z_0) = \lim_{\triangle z\to 0}\frac{f(z_0+\triangle z)-f(z_0)}{\triangle z} = \lim_{\triangle z\to 0}\frac{(z_0+\triangle z)^2-5(z_0+\triangle z)-f(z_0)}{\triangle z}$$
$$= \lim_{\triangle z\to 0}\frac{{z_0}^2+2z_0\triangle z+(\triangle z)^2-5z_0-5\triangle z-({z_0}^2-5z_0)}{\triangle z}$$
$$= \lim_{\triangle z\to 0}\frac{2z_0\triangle z+(\triangle z)^2-5\triangle z}{\triangle z} = \lim_{\triangle z\to 0}\frac{\triangle z(2z_0+\triangle z-5)}{\triangle z} = \lim_{\triangle z\to 0} 2z_0 +\triangle z - 5$$

The limit is $f'(z_0) = 2z_0 - 5$.

Example: Show that if $f(z) = Re(z)$, then $f'(z)$ does not exist.

Solution. $f(z) = Re(z)$

$$f'(z) = \lim_{\triangle z\to 0}\frac{f(z+\triangle z)-f(z)}{\triangle z} = \lim_{\triangle z\to 0}\frac{Re(z+\triangle z)-Re(z)}{\triangle z} = \lim_{\triangle z\to 0}\frac{Re(\triangle z)}{\triangle z}$$

$z = x + iy$, $\triangle z = \triangle x + i\triangle y$

If $\triangle z = \triangle x$, then $f'(z) = \lim_{\triangle x\to 0}\frac{\triangle x}{\triangle x} = 1$

If $\triangle z = i\triangle y$, then $f'(z) = \lim_{\triangle y\to 0}\frac{0}{i\triangle y} = 0 \quad \Longrightarrow f'(z)$ dose not exist.

Differentiation Rules:

Let c be a complex constant, and let f and g be a differentiable function at a point z. Then

e) $\frac{d}{dz}c = 0$,

f) $\frac{d}{dz}z^n = nz^{n-1}, \quad n \in \mathbb{Z}$,

g) $\frac{d}{dz}[cf(z)] = cf'(z)$,

h) $\frac{d}{dz}[f(z) \pm g(z)] = f'(z) \pm g'(z)$,

i) $\frac{d}{dz}[f(z)g(z)] = f'(z)g(z) + f(z)g'(z)$,

j) $\frac{d}{dz}\left[\frac{f(z)}{g(z)}\right] = \frac{f'(z)g(z)-f(z)g'(z)}{[g(z)]^2}$,

k) $\frac{d}{dz}g(f(z)) = g'(f(z))\,f'(z)$.

Example : Find $f'(z)$ if $f(z) = \frac{z^2}{4z+1}$

Solution. $f'(z) = \frac{(4z+1).2z-z^2.4}{(4z+1)^2} = \frac{4z^2+2z}{(4z+1)^2}$.

Theorem: (Differentiability Implies Continuity)

If a function f is differentiable at a point z_0, then f is continuous at z_0.

Proof:

Assume $f'(z_0)$ exists, we want to show that $\lim_{z\to z_0} f(z) = f(z_0)$ or

$$\lim_{z\to z_0}[f(z) - f(z_0)] = 0$$

$$\lim_{z\to z_0}[f(z) - f(z_0)].\frac{(z-z_0)}{(z-z_0)} = f'(z_0).0 = 0$$

Example : $f(z) = |z|^2$ is continuous at each $z \in \mathbb{C}$,

$$f'(z) = \lim_{\triangle z\to 0}\frac{f(z+\triangle z)-f(z)}{\triangle z}$$

$$= \lim_{\triangle z\to 0}\frac{|z+\triangle z|^2-|z|^2}{\triangle z}$$

$z = x + iy$, $\triangle z = \triangle x + i \triangle y$

If $\triangle z = \triangle x$, $z + \triangle z = (x + \triangle x) + iy$

$$f'(z) = \lim_{\triangle x\to 0}\frac{|z+\triangle x|^2-|z|^2}{\triangle x} = \lim_{\triangle x\to 0}\frac{(x+\triangle x)^2+y^2-(x^2+y^2)}{\triangle x}$$

$$= \lim_{\triangle x\to 0}\frac{2x\triangle x+(\triangle x)^2}{\triangle x} = 2x$$

If $\triangle z = i \triangle y$

$$f'(z) = \lim_{\triangle y\to 0}\frac{|z+i\triangle y|^2-|z|^2}{i\triangle y} = \lim_{\triangle y\to 0}\frac{x^2+(y+\triangle y)^2-(x^2+y^2)}{i\triangle y}$$

$$= \lim_{\triangle y\to 0}\frac{2y\triangle y+(\triangle y)^2}{i\triangle y} = \frac{2y}{i} = -2iy$$

If $z \neq 0 \Longrightarrow f'(z)$ does not exist.

The Cauchy- Riemann Equations

Theorem: (Necessary condition for differentiability)

Suppose that $f(z) = u(x,y) + iv(x,y)$ is differentiable at $z_0 = x_0 + iy_0$. Then u_x, u_y, v_x, v_y exist at (x_0, y_0) and satisfy Cauchy-Riemann equations

$$u_x = v_y\,, \quad u_y = -v_x \quad \text{at} \quad (x_0, y_0)$$

Moreover $\quad f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0)$.

Proof. Assume $f'(z_0)$ exists

$$f'(z_0) = \lim_{\triangle z \to 0} \frac{f(z_0+\triangle z) - f(z_0)}{\triangle z}$$

$$\Rightarrow \lim_{\triangle x \to 0} \frac{f(z_0+\triangle x) - f(z_0)}{\triangle x} = f'(z_0) = \lim_{\triangle y \to 0} \frac{f(z_0+i\triangle y) - f(z_0)}{i\triangle y}$$

1) $\quad f'(z_0) = \lim_{\triangle x \to 0} \frac{f(z_0+\triangle x) - f(z_0)}{\triangle x}$

$$= \lim_{\triangle x \to 0} \frac{u(x_0+\triangle x, y_0) + iv(x_0+\triangle x, y_0) - (u(x_0,y_0) + iv(x_0,y_0))}{\triangle x}$$

$$= \lim_{\triangle x \to 0} \frac{u(x_0+\triangle x, y_0) - u(x_0,y_0) + i(v(x_0+\triangle x, y_0) - v(x_0,y_0))}{\triangle x}$$

$$f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0) \qquad (1)$$

2) $\quad f'(z_0) = \lim_{\triangle y \to 0} \frac{f(z_0+i\triangle y) - f(z_0)}{i\triangle y}$

$$= \lim_{\triangle y \to 0} \frac{u(x_0, y_0+\triangle y) + iv(x_0, y_0+\triangle y) - (u(x_0,y_0) + iv(x_0,y_0))}{i\triangle y}$$

$$= \lim_{\triangle y \to 0} \frac{v(x_0, y_0+\triangle y) - v(x_0,y_0) - i(u(x_0, y_0+\triangle y) - u(x_0,y_0))}{\triangle y}$$

$$f'(z_0) = v_y(x_0, y_0) - iu_y(x_0, y_0) \qquad (2)$$

(1) and (2)

$$\Rightarrow \quad u_x(x_0, y_0) = v_y(x_0, y_0)$$
$$v_x(x_0, y_0) = -u_y(x_0, y_0)$$

Remark: The Cauchy-Riemann equations are necessary condition for the differentiability of f at a point z_0. Thus they can be used to find the points at which f is not differentiable.

Example : Show that $f(z) = |z|^2$ is not differentiable at any $z \in \mathbb{C} - \{0\}$.

Solution. $\quad f(z) = |z|^2 = x^2 + y^2$

$$\Rightarrow u(x,y) = x^2 + y^2, \qquad v(x,y) = 0$$

$$u_x = 2x \quad \neq \quad v_y = 0 \quad \text{if } x \neq 0$$
$$u_y = 2y \quad \neq \quad -v_x = 0 \quad \text{if } y \neq 0$$

Thus $u_x = v_y$ and $u_y = -v_x$ only at $(0,0)$

That is the Cauchy-Riemann equations are not satisfied at points $(x, y) \neq (0,0)$ and hence $f'(z)$ does not exist for all $z \neq 0$.

Example : Show that $f(z) = \bar{z}$ is not differentiable at any $z \in \mathbb{C}$.

Solution. $f(z) = \bar{z} = x - iy$

$$\Longrightarrow u(x, y) = x, \quad v(x, y) = -y$$

$$u_x = 1 \quad \neq \quad v_y = -1$$

The Cauchy-Riemann equations are not satisfied at any $(x, y) \in \mathbb{R}^2$

i.e. $f'(z)$ does not exist for any $z \in \mathbb{C}$.

Example : Let $f(z) = \begin{cases} \frac{\bar{z}^2}{z}, & z \neq 0 \\ 0, & z = 0 \end{cases}$. Show that the Cauchy-Riemann equations are satisfied at $(0,0)$, but $f'(0)$ does not exist.

Solution. $f'(0) = \lim\limits_{\triangle z \to 0} \frac{f(\triangle z) - f(0)}{\triangle z} = \lim\limits_{\triangle z \to 0} \frac{\frac{\overline{\triangle z}^2}{\triangle z} - 0}{\triangle z}$

$$= \lim_{\triangle z \to 0} \frac{(\overline{\triangle z})^2}{(\triangle z)^2}$$

If $\triangle z = \triangle x$, then

$$f'(0) = \lim_{\triangle x \to 0} \frac{(\triangle x)^2}{(\triangle x)^2} = 1 \qquad (1)$$

If $\triangle z = \triangle x + i \triangle x$, then

$$f'(0) = \lim_{\triangle x \to 0} \frac{\left(\triangle x(1-i)\right)^2}{\left(\triangle x(1+i)\right)^2} = \frac{(1-i)^2}{(1+i)^2} = \frac{-2i}{+2i} = -1 \qquad (2)$$

(1) and (2) $f'(0)$ does not exist.

$$f(z) = \begin{cases} \frac{\bar{z}^2}{z}, & z \neq 0 \\ 0, & z = 0 \end{cases} = \begin{cases} \frac{(x-iy)^2}{x+iy}, & (x, y) \neq (0,0) \\ 0, & (x, y) = (0,0) \end{cases}$$

$$u(x, y) = \begin{cases} \frac{x^3 - 3xy^2}{x^2 + y^2}, & (x, y) \neq (0,0) \\ 0, & (x, y) = (0,0) \end{cases}$$

$$v(x, y) = \begin{cases} \frac{y^3 - 3x^2 y}{x^2 + y^2}, & (x, y) \neq (0,0) \\ 0, & (x, y) = (0,0) \end{cases}$$

$$u_x(0,0) = \lim_{\triangle x \to 0} \frac{u(\triangle x,0)-u(0,0)}{\triangle x} = \lim_{\triangle x \to 0} \frac{\frac{(\triangle x)^3}{(\triangle x)^2}-0}{\triangle x} = 1$$

$$v_y(0,0) = \lim_{\triangle y \to 0} \frac{v(0,\triangle y)-v(0,0)}{\triangle y} = \lim_{\triangle y \to 0} \frac{\frac{(\triangle y)^3}{(\triangle y)^2}-0}{\triangle y} = 1$$

$$u_y(0,0) = \lim_{\triangle y \to 0} \frac{u(0,\triangle y)-u(0,0)}{\triangle y} = \lim_{\triangle y \to 0} \frac{0-0}{\triangle y} = 0$$

$$v_x(0,0) = \lim_{\triangle x \to 0} \frac{v(\triangle x,0)-v(0,0)}{\triangle x} = 0 .$$

Theorem: (Sufficient condition for differentiability)

Let $f(z) = u(x,y) + iv(x,y)$ be defined throughout some $\varepsilon -$ neighborhood $N_\varepsilon(z_0)$ of a point $z_0 = x_0 + iy_0$.
Suppose that u_x, u_y, v_x, v_y exist everywhere in $N_\varepsilon(z_0)$ and they are continuous at (x_0, y_0). Then if

$$u_x = v_y \,, \quad u_y = -v_x \quad \text{at} \quad (x_0, y_0)$$

Then $f'(z_0)$ exists.

Example : Let $f(z) = \sin x \cosh y + i \cos x \sinh y$.
Show that $f'(z)$ exists for all $z \in \mathbb{C}$ and find it.
Solution.

$$u(x,y) = \sin x \cosh y \,, \qquad v(x,y) = \cos x \sinh y$$

$$u_x = \cos x \cosh y \;=\; v_y = \cos x \cosh y$$

$$u_y = \sin x \sinh y \;=\; -v_x = -\sin x \sinh y$$

u_x, u_y, v_x, v_y exists and continuous for all $(x,y) \in \mathbb{R}^2$.
And the Cauchy-Riemann equations are satisfied at any $(x,y) \in \mathbb{R}^2$
Thus $f'(z)$ exists for any $z \in \mathbb{C}$.

$$f'(z) = u_x(x,y) + iv_x(x,y)$$

$$= \cos x \cosh y - i \sin x \sinh y.$$

Exercise : Find the value of the real numbers a and b that make
$f(z) = x^2 + ay^2 + e^x \cos y + i[bxy + e^x \sin y]$.
has derivative at every $z \in \mathbb{C}$.

Polar coordinates

For $z \neq 0$, we have $z = re^{i\theta}$ and hence $f(z) = u(r,\theta) + iv(r,\theta)$.
Using the chain rule, we find:

$u_r = u_x \cos\theta + u_y \sin\theta,$ $\quad v_r = v_x \cos\theta + v_y \sin\theta$

$u_\theta = -u_x r \sin\theta + u_y r \cos\theta,$ $\quad v_\theta = -v_x r \sin\theta + v_y r \cos\theta$

If u and v satisfy the Cauchy Riemann equations, then

$u_r = \frac{1}{r} v_\theta$ and $v_r = -\frac{1}{r} u_\theta$

Remark : If $f'(z_0)$ exists, then $f'(z_0) = e^{-i\theta_0}[u_r(r_0,\theta_0) + iv_r(r_0,\theta_0)]$
Proof: يوضح اثناء المحاضرة

Example : Show that $f(z) = \frac{1}{z}$ is differentiable at each $z \in \mathbb{C} - \{0\}$ and $f'(z) = -\frac{1}{z^2}$

Solution: Let $z \neq 0$, $z = re^{i\theta}$,

$f(z) = \frac{1}{z} = \frac{1}{re^{i\theta}} = \frac{1}{r} e^{-i\theta} = \frac{1}{r}(\cos\theta - i\sin\theta)$

$u(r,\theta) = \frac{1}{r}\cos\theta$

$v(r,\theta) = -\frac{1}{r}\sin\theta$

$u_r = \frac{-1}{r^2}\cos\theta$ $\quad v_\theta = \frac{-1}{r}\cos\theta$

$v_r = \frac{1}{r^2}\sin\theta$ $\quad u_\theta = \frac{-1}{r}\sin\theta$

$u_r = \frac{1}{r} v_\theta$ and $v_r = -\frac{1}{r} u_\theta$

Therefore, $f'(z)$ exists $\forall z \in \mathbb{C} - \{0\}$

$$f'(z) = e^{-i\theta}[u_r(r,\theta) + iv_r(r,\theta)]$$
$$= e^{-i\theta}\left[\frac{-1}{r^2}\cos\theta + i\frac{1}{r^2}\sin\theta\right]$$
$$= \frac{-1}{r^2} e^{-i\theta}[\cos\theta - i\sin\theta]$$
$$= \frac{-1}{r^2} e^{-i\theta} e^{-i\theta} = \frac{-1}{r^2} e^{-2i\theta} = -\frac{1}{z^2}$$

Analytic Functions

Definition: A complex function $w = f(z)$ is said to be **analytic** at a point z_0 if f is differentiable at z_0 and at every point in some neighborhood of z_0.

A function $\boldsymbol{f}$ **is analytic in a domain D** if it is analytic at every point in D.

Example : Show that $f(z) = 1/z$ is analytic at $z = 1$.

Solution. $f(z) = \frac{1}{z}$, $z_0 = 1$

$f'(z) = \frac{-1}{z^2}$ exists for all $z \neq 0$.

In particular $f'(z)$ exists $\forall z \in \left\{z\colon\ |z-1| < \frac{1}{2}\right\} = N_{\frac{1}{2}}(1)$

$\Longrightarrow f$ is analytic at $z = 1$.

Example : Show that $f(z) = 1/(z-2)^2$ is analytic in $S = \{z\colon\ |z| < 1\}$.

Solution. S is open and connected (S is domain)

$\Longrightarrow$ f is analytic in S if and only if $f'(z)$ exists $\forall z \in S$.

$f'(z) = \frac{-2}{(z-2)^3}$, exists everywhere except 2

$\forall z \in \mathbb{C} - \{2\}$

Thus $f'(z)$ exists $\forall z \in S$.

Example : Let $f(z) = |z|^2$. Show that

a. $f'(0) = 0$.
b. $f(z)$ is not analytic at $z = 0$.

Solution.

a. $f'(0) = 0?$

$f(z) = x^2 + y^2$

$u(x, y) = x^2 + y^2, \quad v(x, y) = 0$

$u_x = 2x, \quad u_y = 2y, \ v_x = 0, \ v_y = 0$

u_x, u_y, v_x, v_y exists and continuous for all $(x, y) \in \mathbb{R}^2$.

And the Cauchy-Riemann equations are satisfied at (0,0)

$$u_x(0,0) = 0 = v_y(0,0)$$
$$u_y(0,0) = 0 = -v_x(0,0)$$

Thus $f'(z)$ exists for $z = 0$.

$$f'(0) = u_x(0,0) + iv_x(0,0) = 0$$

b. Since $f'(z)$ does not exist $\forall z \neq 0, f'(z)$ does not exist in any ε –Neighborhood of $z = 0$ and hence f is not analytic at $z = 0$.

Remark: A function f is analytic in a set S which is not open, it is to be understood that f is analytic in an open set containing S.

Example: $f(z) = 1/z$ is analytic in the closed set $S = \left\{z\colon |z-1| \leq \frac{1}{2}\right\}$ since it is analytic in the open set $\{z\colon |z-1| < 1\} \supseteq S$.

Definition: (Entire functions)

A function that is analytic at every point z in the complex plane is said to be an **entire function**.

Example: Every polynomial $f(z) = a_0 + a_1 z + \cdots + a_n z^n$ is an entire function.

Definition: (Singular point)

If a function f is not analytic at a point z_0 but it is analytic at some point in every neighborhood of z_0, then z_0 is called singular point of f.

Remark: If $f(z) = \frac{P(z)}{Q(z)}$ is a rational function, then the only singular points of f are z such that $Q(z) = 0$.

Example: Find the singular points of the following functions:

a) $f(z) = 1/z^2$,
b) $f(z) = \frac{z}{z^2+1}$,
c) $f(z) = |z|^2$.

Solution.

a) Singular points are z such that $z^2 = 0$
$\Longrightarrow z = 0$.

b) Singular points are z such that $z^2 + 1 = 0$
$\Longrightarrow z = \mp i$.
c) Has no singular point since f is nowhere analytic.

Remark: Let D be a domain.

1) If a function f is analytic in D, then f is continuous in D.
2) If $f = u + iv$ is analytic in D, then u and v satisfy the Cauchy- Riemann equations.
3) If u_x, u_y, v_x, v_y are continuous on D and $u_x = v_y,\ u_y = -v_x$, then f is analytic on D.
4) If f and g are analytic in D, then $f \mp g, fg$ are analytic in D and f/g is analytic provided that $g(z) \neq 0$ for all $z \in D$.
5) A composition of two analytic function is analytic.

Exercises:

1) Show that $f = ze^x e^{yi}$ is an entire function.
2) Show that $f = e^y e^{2xi}$ is nowhere analytic.
3) Show that if $f(z) = u + iv$ and $\overline{f(z)} = u - iv$ are both analytic in a domain D, then $f(z) =$ constant.
4) Find real constants a, b, c, and d so that the given function is analytic $f(z) = x^2 + axy + by^2 + i(cx^2 + dxy + y^2)$.

Chapter 3: Elementary Functions

1. Complex Exponential Function

We have Euler's formula: $e^{i\theta} = \cos\theta + i\sin\theta$. We can extend to the complex exponential function e^z.

Definition: For $z = x + iy$ the **complex exponential function** is defined as

$e^z = e^{x+yi} = e^x e^{iy} = e^x(\cos y + i\sin y)$

Remark:

1. Domain $= all\ of\ \mathbb{C}$.
2. We sometimes use $exp(z)$ to denote e^z.

Example : Express $e^{-1+\frac{\pi}{2}i}$ in the form $a + ib$.

Solution: $e^{-1+\frac{\pi}{2}i} = exp\left(-1 + \frac{\pi}{2}i\right) = e^{-1}\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) = \frac{i}{e}$.

Example : Show that e^z is entire function and

$\frac{d}{dz}e^z = e^z$.

Solution:

We have $e^z = e^x(\cos y + i\sin y)$ if $z = x + iy$.

Thus, $u(x,y) = e^x \cos y$ and $v(x,y) = e^x \sin y$.

Differentiating u with respect to x and y, we find

$u_x = e^x \cos y$, $u_y = -e^x \sin y$.

Differentiating v with respect to x and y, we find

$v_x = e^x \sin y$, $v_y = e^x \cos y$.

We see that $u_x = v_y$ and $u_y = -v_x$. Hence the Cauchy Riemann equations are satisfied at all points. Moreover, the functions u_x, v_x, u_y, v_y are continuous.

Therefore, e^z is analytic at all points, or entire.

$\frac{d}{dz}e^z = u_x + i\,v_x = e^x \cos y + ie^x \sin y = e^z$.

Remark: When $z = \frac{1}{n}$, $n = 2,3,\ldots$, then $e^{\frac{1}{n}} = \sqrt[n]{e}$ that is $e^{\frac{1}{n}}$ is not the set of all n-th roots of e.

Proposition: (properties of e^z)

Let $z = x + iy$. Then we have the following properties of e^z:

1. $|e^z| = e^x$.
2. $arg(e^z) = y + 2n\pi, \ \ n \in \mathbb{Z}$.
3. $e^z \neq 0$ for all $z \in \mathbb{C}$.
4. $e^0 = 1$.
5. $e^{z_1}e^{z_2} = e^{z_1+z_2}$.
6. $\frac{1}{e^z} = e^{-z}$.
7. $\frac{e^{z_1}}{e^{z_2}} = e^{z_1-z_2}$.
8. $(e^z)^n = e^{nz}$ for all $n \in \mathbb{Z}$.
9. e^z is periodic function with period of $2\pi i$, that is $e^{z+2\pi i} = e^z$.

Example: Solve the equation $e^z = 1$.

Solution: $e^z = e^x(\cos y + i\sin y) = 1$

$e^x \cos y = 1$

and, $e^x \sin y = 0 \ \Rightarrow \ \ \sin y = 0 \ \Rightarrow y = n\pi, \ \ \ n \in \mathbb{Z}$.

$\Rightarrow \ e^x \cos(n\pi) = 1$

$e^x(-1)^n = 1 \ \Rightarrow \ \ e^x = (-1)^n$

since $e^x > 0$, then $n = 2m, \ m \in \mathbb{Z}$.

$e^x = 1 \ \ \ \Rightarrow x = 0, \ \ y = 2m\pi, \ \ m \in \mathbb{Z}$.

$z = 2m\pi i, \ \ m \in \mathbb{Z}$.

Example: Find all the value of z such that $e^{iz} = -2$.

Solution: $-2 = |-2|e^{i(Arg(-2)+2n\pi)} = 2e^{i(\pi+2n\pi)}, \ \ n \in \mathbb{Z}$

$e^{iz} = -2$

$e^{ix-y} = 2e^{i(\pi+2n\pi)}$

$e^{-y}e^{xi} = 2e^{i(\pi+2n\pi)}$

$\Rightarrow \ e^{-y} = 2$ and $x = \pi + 2n\pi, \ \ n \in \mathbb{Z}$

$y = -ln2$

Therefor, $z = (2n+1)\pi - i\,ln2, \ \ n \in \mathbb{Z}$.

Exercises:

1. Find the derivative of each of the following functions:
   a) $iz^4(z^2 - e^z)$.
   b) $e^{z^2-(1+i)z+3}$.
2. Prove that $e^{\bar{z}} = \overline{e^z}$
3. Solve the equations:
   a) $e^z = 1 + \sqrt{3}i$
   b) $e^{2z-1} = 1$

2. Trigonometric Functions

Euler's formula gives: $e^{ix} = \cos x + i \sin x$

and $\quad e^{-ix} = \cos x - i \sin x$

thus we have

$$\cos x = \frac{1}{2}\left(e^{ix} + e^{-ix}\right)$$

and

$$\sin x = \frac{1}{2i}\left(e^{ix} - e^{-ix}\right)$$

Definition: The complex **sine** and **cosine** functions is defined by:

$$\sin z = \frac{1}{2i}\left(e^{iz} - e^{-iz}\right) \quad \text{and} \quad \cos z = \frac{1}{2}\left(e^{iz} + e^{-iz}\right)$$

Definition: We define the complex **tangent**, **cotangent**, **secant** and **cosecant** functions using the complex sine and cosine:

$$\tan z = \frac{\sin z}{\cos z}, \quad \cot z = \frac{\cos z}{\sin z}, \quad \sec z = \frac{1}{\cos z} \quad \text{and} \quad \csc z = \frac{1}{\sin z}.$$

Example: Express the value of the given trigonometric function in the form $a + ib$.

(a) $\sin i\pi$ (b) $\cos i$ (c) $\tan(\pi - 2i)$

Solution:

(a) $\sin i\pi = \frac{1}{2i}\left(e^{i(i\pi)} - e^{-i(i\pi)}\right) = \frac{-i}{2}\left(e^{-\pi} - e^{\pi}\right) = i\left(\frac{e^{\pi} - e^{-\pi}}{2}\right) = i \sinh \pi$

(b) $\cos i = \frac{1}{2}\left(e^{i.i} + e^{-i.i}\right) = \frac{e^{-1} + e}{2} = \cosh 1$

(c) $$\tan(\pi - 2i) = \frac{\left(e^{i(\pi-2i)} - e^{-i(\pi-2i)}\right)/2i}{\left(e^{i(\pi-2i)} + e^{-i(\pi-2i)}\right)/2} = \frac{e^{i(\pi-2i)} - e^{-i(\pi-2i)}}{\left(e^{i(\pi-2i)} + e^{-i(\pi-2i)}\right)i}$$
$$= -i\,\frac{e^{i\pi}e^{2} - e^{-i\pi}e^{-2}}{e^{i\pi}e^{2} + e^{-i\pi}e^{-2}} = -i\,\frac{e^{2} - e^{-2}}{e^{2} + e^{-2}}.$$

Example: Show that $\sin(z + 2n\pi) = \sin z$.

Solution: $$\sin(z + 2n\pi) = \frac{e^{i(z+2n\pi)} - e^{-i(z+2n\pi)}}{2i} = \frac{e^{iz}e^{2n\pi i} - e^{-iz}e^{-2n\pi i}}{2i}$$
$$= \frac{e^{iz} - e^{-iz}}{2i} = \sin z.$$

Remark:

1) The sine and cosine functions are entire functions.
2) $\frac{d}{dz}\sin z = \cos z$ and $\frac{d}{dz}\cos z = -\sin z$.
3) $\sin(-z) = -\sin z$ and $\cos(-z) = \cos z$.
4) $\sin z = 0$ if and only if $z = n\pi,\ n \in \mathbb{Z}$.
5) $\cos z = 0$ if and only if $z = (2n+1)\frac{\pi}{2},\ n \in \mathbb{Z}$.
6) $\tan z$ and $\sec z$ are analytic everywhere except at the singular points $z = (2n+1)\frac{\pi}{2},\ n \in \mathbb{Z}$.
7) $\cot z$ and $\csc z$ are analytic everywhere except at the singular points $z = n\pi,\ n \in \mathbb{Z}$.
8) $\frac{d}{dz}\tan z = \sec^2 z$, $\quad \frac{d}{dz}\cot z = -\csc^2 z$

   $\frac{d}{dz}\sec z = \sec z \tan z$, $\quad \frac{d}{dz}\csc z = -\csc z \cot z$.

Proof: 8) $\frac{d}{dz}\tan z = \sec^2 z$.

$$f(z) = \tan z = \frac{\sin z}{\cos z} = \frac{(e^{iz}-e^{-iz})/2i}{(e^{iz}+e^{-iz})/2} = \frac{(e^{iz}-e^{-iz})/i}{e^{iz}+e^{-iz}}$$

$$f'(z) = \frac{(e^{iz}+e^{-iz})(e^{iz}+e^{-iz}) - \frac{1}{i}(e^{iz}-e^{-iz})(ie^{iz}-ie^{-iz})}{\left(e^{iz}+e^{-iz}\right)^2}$$

$$= \frac{e^{2iz}+2+e^{-2iz}-\left(e^{2iz}-2+e^{-2iz}\right)}{\left(e^{iz}+e^{-iz}\right)^2} = \frac{4}{\left(e^{iz}+e^{-iz}\right)^2} = \left(\frac{2}{e^{iz}+e^{-iz}}\right)^2$$

$$= \left(\frac{1}{\frac{e^{iz}+e^{-iz}}{2}}\right)^2 = \left(\frac{1}{\cos z}\right)^2 = sec^2 z.$$

Proposition: (Trigonometric identities)

Let z, z_1 and z_2 be complex numbers. Then:

1) $\sin^2 z + \cos^2 z = 1$.
2) $\sin(z_1 \pm z_2) = \sin z_1 \cos z_2 \pm \sin z_2 \cos z_1$.
3) $\cos(z_1 \pm z_2) = \cos z_1 \cos z_2 \mp \sin z_1 \sin z_2$.
4) If $z = iy$ is pure imaginary, then

   $\sin(iy) = i \sinh y$ and $\cos(iy) = \cosh y$.
5) If $z = x + iy$, then

   $\cos z = \cos x \cosh y - i \sin x \sinh y$.
6) If $z = x + iy$, then

   $\sin z = \sin x \cosh y + i \cos x \sinh y$.
7) If $z = x + iy$, then

   $|\sin z|^2 = \sin^2 x + \sinh^2 y$

   $|\cos z|^2 = \cos^2 x + \sinh^2 y$

Example: Find all the roots of $\sin z = 1$.

Solution: Since $\sin z = \sin x \cosh y + i \cos x \sinh y = 1$.
Then $\sin x \cosh y = 1$,
And $\cos x \sinh y = 0 \quad \Rightarrow \cos x = 0$ or $\sinh y = 0$

$$\Rightarrow x = (2n+1)\frac{\pi}{2},\ n \in \mathbb{Z} \quad \text{or} \quad y = 0.$$

Therefor $\sin(2n+1)\frac{\pi}{2}\cosh y = 1$

$$(-1)^n \cosh y = 1$$

$$\cosh y = (-1)^n$$

Since $\cosh y \geq 1 \quad \Rightarrow \quad n = 2m,\ m \in \mathbb{Z}$
$\cosh y = 1 \quad \Rightarrow y = 0$
The solutions are:

$$z = (4m+1)\frac{\pi}{2},\ \ m \in \mathbb{Z}$$

If $y = 0 \quad \Rightarrow \sin x = 1$
Thus we have the same solutions $z = (4m+1)\frac{\pi}{2},\ \ m \in \mathbb{Z}$.

Example: Show that $\overline{\cos(iz)} = \cos(i\bar{z})$.

Solution: $\cos(iz) = \frac{1}{2}\left[e^{i(iz)} + e^{-i(iz)}\right] = \frac{1}{2}\left[e^{-z} + e^{z}\right]$

$\overline{\cos(iz)} = \overline{\frac{1}{2}\left[e^{-z} + e^{z}\right]}$

$$= \frac{1}{2}\left[\overline{e^{-z}} + \overline{e^{z}}\right] = \frac{1}{2}\left[e^{-\bar{z}} + e^{\bar{z}}\right] \ \dots (1)$$

$\cos(i\bar{z}) = \frac{1}{2}\left[e^{i(i\bar{z})} + e^{-i(i\bar{z})}\right] = \frac{1}{2}\left[e^{-\bar{z}} + e^{\bar{z}}\right] \ \dots (2)$

From (1) and (2) $\Rightarrow \overline{\cos(iz)} = \cos(i\bar{z})$.

Example: Find all the roots of $\sin z = \cos z$.

Solution: $\sin z = \sin x \cosh y + i \cos x \sinh y$
$\cos z = \cos x \cosh y - i \sin x \sinh y$
$\sin x \cosh y = \cos x \cosh y \ \dots (1)$
$\cos x \sinh y = -\sin x \sinh y \ \dots (2)$
From (1) $\Rightarrow$
$\cosh y (\sin x - \cos x) = 0$
$\cosh y \neq 0$ since $\cosh y \geq 1$
$\Rightarrow \sin x = \cos x$
$\cos x = 0 \Rightarrow \sin x = 0 \Rightarrow x = (2n+1)\frac{\pi}{2} = n\pi$ impossible.

$\cos x \neq 0 \;\Rightarrow\; \tan x = 1$
$\Rightarrow\; x = \frac{\pi}{4} + n\pi, \quad n \in \mathbb{Z}.$
From (2) $\Rightarrow$
$\sinh y\,(\cos x + \sin x) = 0$
$\sinh y\left(\cos\left(\frac{\pi}{4} + n\pi\right) + \sin\left(\frac{\pi}{4} + n\pi\right)\right) = 0$
$\Rightarrow\; \sinh y\left(\frac{(-1)^n}{\sqrt{2}} + \frac{(-1)^n}{\sqrt{2}}\right) = 0$
$\sinh y = 0 \;\Rightarrow y = 0$
Thus, the solutions are:
$z = \frac{\pi}{4} + n\pi,\; n \in \mathbb{Z}.$

<u>Exercises</u>:

1. Express the value, $\sec\left(\frac{\pi}{2} - i\right)$ in the form $a + ib$.
2. Verify the given trigonometric identity
   $\cos(z_1 + z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2.$
3. Find the derivative of the given function:
   a) $\sin(z^2)$
   b) $z \tan\frac{1}{z}$

3. Hyperbolic Functions.

Definition: We define the hyperbolic functions of a complex variable z as follows:

$\sinh z = \frac{1}{2}[e^z - e^{-z}]$

$\cosh z = \frac{1}{2}[e^z + e^{-z}]$

and $\tanh z = \frac{\sinh z}{\cosh z}$, $\coth z = \frac{\cosh z}{\sinh z}$

$\operatorname{sech} z = \frac{1}{\cosh z}$, $\operatorname{csch} z = \frac{1}{\sinh z}$.

Proposition:

1. $\cosh z = \cos(iz)$ and $\sinh z = -i\sin(iz)$.
2. $\cosh z$ and $\sinh z$ are entire functions.
3. $\frac{d}{dz}\sinh z = \cosh z$ and $\frac{d}{dz}\cosh z = \sinh z$.
4. $\sinh(-z) = -\sinh z$ and $\cosh(-z) = \cosh z$.
5. $\cosh^2 z - \sinh^2 z = 1$.
6. $\sinh(z_1 + z_2) = \sinh z_1 \cosh z_2 + \sinh z_2 \cosh z_1$.
7. $\cosh(z_1 + z_2) = \cosh z_1 \cosh z_2 + \sinh z_1 \sinh z_2$.
8. $|\sinh z|^2 = \sinh^2 x + \sin^2 y$.
9. $|\cosh z|^2 = \sinh^2 x + \cos^2 y$.
10. $\sinh z = 0$ if and only if $z = n\pi i, n \in \mathbb{Z}$.
11. $\cosh z = 0$ if and only if $z = (2n+1)\frac{\pi i}{2}, n \in \mathbb{Z}$.

Example: Show that $\sinh(z + i\pi) = -\sinh z$?

Solution: $\sinh(z + i\pi) = -i\sin(iz - \pi)$

$= -i[\sin(iz)\cos\pi - \sin\pi\cos(iz)]$

$= i\sin(iz)$

$= -\sinh z$.

Example: Find all the roots of $\cosh z = i$.

Solution: $\cosh z = \cos(iz) = i$

$\Rightarrow \cos(-y + ix) = i$

$\Rightarrow \cos y \cosh x - i\sin(-y)\sinh x = i$

$\Rightarrow \cos y \cosh x + i\sin y \sinh x = i$

$\Rightarrow \cos y \cosh x = 0 \;\dots\; (1)$

and $\sin y \sinh x = 1 \;\dots\; (2)$

from (1) $\Longrightarrow$ $\cos y = 0 \Longrightarrow y = (2n+1)\frac{\pi}{2}\ ,\ n \in \mathbb{Z}$.

Therefor,

$\sin(2n+1)\frac{\pi}{2}\sinh x = 1$

$(-1)^n \sinh x = 1$

$\sinh x = (-1)^n$

$\Longrightarrow \frac{1}{2}[e^x - e^{-x}] = (-1)^n$

$\Longrightarrow\ e^{2x} - 1 = 2(-1)^n e^x$

$(e^x)^2 - 2(-1)^n e^x - 1 = 0$

$e^x = \frac{2(-1)^n \mp \sqrt{4+4}}{2}$

$\quad = (-1)^n \mp \sqrt{2}$

Since $e^x > 0\ \Longrightarrow e^x = (-1)^n + \sqrt{2}$

$\qquad\qquad \Longrightarrow\ x = \ln\left[(-1)^n + \sqrt{2}\right]$

The solution are $z = \ln\left[(-1)^n + \sqrt{2}\right] + (2n+1)\frac{\pi}{2}i\ ,\ n \in \mathbb{Z}$.

Exercises:

1. Express the value, $\cosh\left(1 + \frac{\pi}{6}i\right)$ in the form $a + ib$.
2. Verify the given hyperbolic identity
   $Im(\cosh z) = \sinh x \sin y$
3. Find the derivative of the given function:
   a) $\sin z \sinh z$
   b) $\tanh(iz - 2)$.

4. The Logarithmic Function and its Branches

Definition: (The logarithm of a complex number)

We define the logarithm of a nonzero complex number z to be

$\log z = \ln|z| + i(Arg\ z + 2n\pi), n \in \mathbb{Z}$

or

$\log z = \ln|z| + i \arg z$

Example: Find **1)** $\log(-1)$, **2)** $\log(1+i)$, **3)** $\log(e^z)$

Solution:

1) $\log(-1) = \ln|-1| + i\arg(-1)$

$= i(\pi + 2n\pi) = (2n+1)\pi i, n \in \mathbb{Z}$

2) $\log(1+i) = \ln|1+i| + i\arg(1+i)$

$|1+i| = \sqrt{2}$ $\quad arg(1+i) = Arg(1+i) + 2n\pi$

$\Rightarrow \log(1+i) = \ln\sqrt{2} + \left(2n + \frac{1}{4}\right)\pi i, n \in \mathbb{Z}$

3) $\log(e^z) = \ln|e^z| + i\arg(e^z)$

$= \ln|e^x| + i(y + 2n\pi), n \in \mathbb{Z}$

$= x + iy + 2n\pi i, n \in \mathbb{Z}$

$= z + 2n\pi i, n \in \mathbb{Z}$

Remark:

1. $\log z$ is not function. We will call it a multiple- valued function.
2. $e^{\log z} = z$ for all $z \neq 0$.
3. $\log e^z = z + 2n\pi i,\ n \in \mathbb{Z}$ for all $z \in \mathbb{C}$.

Definition: The principal value of the logarithm of a nonzero complex number $z, \log z$, is defined to be

$$\operatorname{Log} z = \ln|z| + iArg\ z \qquad -\pi < Arg\ z \leq \pi$$

Example: $\operatorname{Log}(1+\sqrt{3}i) = \ln|1+\sqrt{3}i| + iArg\,(1+\sqrt{3}i)$

$= \ln 2 + i\tan^{-1}\left(\frac{\sqrt{3}}{1}\right)$

$= \ln 2 + i\,\frac{\pi}{3}$

Remark:

1. $\operatorname{Log} z$ is a function.
2. $\log z = \operatorname{Log} z + 2n\pi i\ ,\ n \in \mathbb{Z}$

Example: Show that:

1. $\operatorname{Log}(-ei) = 1 - \frac{\pi}{2} i$

   Solution:

   $$\operatorname{Log}(-ei) = \ln|-ei| + iArg\,(-ei)$$
   $$= \ln e + i\left(-\frac{\pi}{2}\right) = 1 - \frac{\pi}{2} i$$

2. $\operatorname{Log}(1-i) = \frac{1}{2}\ln 2 - \frac{\pi}{4} i$

   Solution:

   $$\operatorname{Log}(1-i) = \ln|1-i| + iArg\,(1-i)$$
   $$= \ln\sqrt{2} + i\left(-\frac{\pi}{4}\right) = \frac{1}{2}\ln 2 - \frac{\pi}{4} i$$

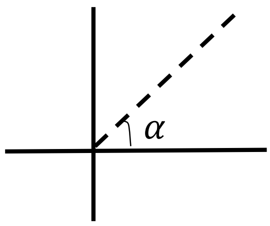
Definition: Let f be a multiple- valued function. A single- valued function F is called a branch of f if there is a domain D such that:

1. $F(z) \in f(z) \quad \forall z \in D$
2. F is analytic in D

Example: $\operatorname{Log} z = \ln|z| + iArg\ z\ , \quad |z| > 0,\ -\pi < Arg\ z < \pi$

Is a branch of $\log z$. It is called the principal branch.

Its derivative is $\frac{1}{z}$



Example: $z \neq 0\ , \quad z = re^{i\theta}, r = |z| > 0$

Let α real ,

$\mathrm{f(z)} = \log z = \ln|z| + i\theta, \quad |z| > 0, \quad \alpha < \theta < \alpha + 2\pi$

Is a branch of $\log z$ and its derivative is given by: $\mathrm{f}'(\mathrm{z}) = \frac{1}{z}$

Proposition: Let z_1 and z_2 be two nonzero complex numbers. Then:

1. $\log(z_1 z_2) = \log z_1 + \log z_2$
2. $\log\left(\frac{z_1}{z_2}\right) = \log z_1 - \log z_2$
3. $\log {z_1}^n = \mathrm{n} \log z_1$.

Exercises:

1. Let $z_1 = -1$ and $z_2 = i$. Verify that $\log(z_1 z_2) = \log z_1 + \log z_2$
2. Let z be a nonzero complex number, show that $z^n = e^{n \log z}, n \in \mathbb{Z}$ and $z^{\frac{1}{n}} = e^{\frac{1}{n}\log z}, n \in \mathbb{N}$
3. Show that $\operatorname{Log}(i^3) \neq 3 \log i$.

5. Complex Exponents

Definition:

Let z be a nonzero complex number and let c be any complex number. We define z^c as:

$$z^c = e^{c \log z}$$

Where

$$\log z = \ln|z| + i \arg z$$

Example: Find the value of 1^i

Solution:

$$1^i = e^{i \log 1} = e^{i(2\pi n i)} = e^{-2n\pi}, \ n \in \mathbb{Z}$$

Example: Find the value of i^{2+i}

Solution:

$$\begin{aligned} i^{2+i} = e^{(2+i)\log i} &= e^{(2+i)\,[\ln|i| + i \arg i]} \\ &= e^{(2+i)\left[\mathrm{i}\left(\frac{\pi}{2}+2n\pi\right)\right]}, \ n \in \mathbb{Z} \\ &= e^{(4n+1)\pi\mathrm{i} - \left(2\mathrm{n}+\frac{1}{2}\right)\pi} \\ &= e^{-\left(2\mathrm{n}+\frac{1}{2}\right)\pi}\, e^{(4n+1)\pi\mathrm{i}} \\ &= e^{-\left(2\mathrm{n}+\frac{1}{2}\right)\pi}\, e^{4n\pi\mathrm{i}}\, e^{\pi i} = -e^{-\left(2\mathrm{n}+\frac{1}{2}\right)\pi}, \ n \in \mathbb{Z} \end{aligned}$$

Example: Find the value of $\left(1+\sqrt{3}i\right)^{3/2}$

Solution:

$$\begin{aligned} \left(1+\sqrt{3}i\right)^{3/2} &= e^{\frac{3}{2}\log\left(1+\sqrt{3}i\right)} \\ &= e^{\frac{3}{2}\left[\ln\left|1+\sqrt{3}i\right| + i \arg\left(1+\sqrt{3}i\right)\right]} \\ &= e^{\frac{3}{2}\left[\ln 2 + i\left(\frac{\pi}{3}+2n\pi\right)\right]}, \ n \in \mathbb{Z} \\ &= e^{\frac{3}{2}\ln 2}\, e^{i\left(\frac{\pi}{2}+3n\pi\right)} \\ &= \sqrt{8}\, e^{i\frac{\pi}{2}}\, e^{3n\pi i} = \pm i\sqrt{8} \end{aligned}$$

Remarks:

1. In general z^c is a multiple-valued function.
2. If $z \neq 0$ and c be any complex number, then $\frac{1}{z^c} = z^{-c}$.

Proposition: Let α be real number and let

$$\log z = \ln|z| + i \arg z, \quad |z| > 0, \ \alpha < \arg z < \alpha + 2\pi$$

Be a branch of $\log z$. Then $z^c = e^{c \log z}$ is analytic in

$D = \{(r,\theta): r > 0,\ \ \alpha < \theta < \ \alpha + 2\pi\}$. moreover $\frac{d}{dz} z^c = cz^{c-1}$

Proof: $\log z = \ln|z| + i \arg z$ is analytic in D
$D = \{z: |z| > 0,\ \alpha < \arg z < \ \alpha + 2\pi\ \}$
e^z is an entire function
$\Rightarrow\ z^c = e^{c \log z}$ is analytic in D

$$\frac{d}{dz} z^c = \frac{d}{dz} e^{c \log z}$$
$$= e^{c \log z} \frac{d}{dz} c \log z$$
$$= e^{c \log z}\, c \frac{1}{z}$$
$$= z^c\ \frac{c}{z} = cz^{c-1}.$$

Definition: The principal value of z^c is $z^c = e^{c \operatorname{Log} z}$.

Example: Find the principal value of $(1+i)^{2+3i}$

Solution: The principal value of

$$(1+i)^{2+3i} = e^{(2+3i)\operatorname{Log}(1+i)}$$
$$= e^{(2+3i)\,[\ln|1+i| + i \operatorname{Arg}(1+i)]}$$
$$= e^{(2+3i)\left[\ln\sqrt{2} + i\frac{\pi}{4}\right]}$$
$$= e^{2\ln\sqrt{2} + i\frac{\pi}{2} + 3i\ln\sqrt{2} - \frac{3\pi}{4}}$$
$$= e^{2\ln\sqrt{2}}\ e^{i\frac{\pi}{2}}\ e^{3i\ln\sqrt{2}}\ e^{-\frac{3\pi}{4}}$$
$$= 2ie^{-\frac{3\pi}{4}}\left[\cos 3\ln\sqrt{2} + i \sin 3\ln\sqrt{2}\right]$$

Example: Find the derivative of the principal value z^i at the point $z = 1 + i$.

Solution:

Because the point $z = 1 + i$ is in the domain $|z| > 0, -\pi < \arg z < \ \pi$

$$\frac{d}{dz}(1+i)^i = i(1+i)^{i-1}$$
$$= i(1+i)^i(1+i)^{-1} = i(1+i)^i \frac{1}{1+i} = \frac{1+i}{2}(1+i)^i$$

Exercises:

1. Find the values of the complex power: (a) i^{2i} (b) $(1+i)^i$
2. Find the principal value of each complex power: (a) $(-3)^{i/\pi}$ (b) $(2i)^{1-i}$.
3. Find the derivative of the principal value z^{1+i} at the point $z = 1 + \sqrt{3}i$.

Chapter 4: Integration in the complex plane

1. Definite Integrals

Definition: If u and v are real-valued function of a real variable t, continuous on a common interval $a \le t \le b$, then we define **the integral** of the complex-valued function $w(t) = u(t) + iv(t)$ on $a \le t \le b$:

$$\int_a^b w(t)dt = \int_a^b u(t)dt + i\int_a^b v(t)dt$$

The continuity of u and v on $[a, b]$ guarantees that both $\int_a^b u(t)dt$ and $\int_a^b v(t)dt$ exist.

Thus $Re\int_a^b w(t)dt = \int_a^b Re\big(w(t)\big)dt$ and $Im\int_a^b w(t)dt = \int_a^b Im\big(w(t)\big)dt.$

<u>Example</u>: Evaluate $\int_0^1 (1+it)^2 dt$.

Solution:

$$\int_0^1 (1+it)^2 dt = \int_0^1 (1+2it-t^2)dt$$
$$= \int_0^1 (1-t^2)dt + i\int_0^1 2tdt = \frac{2}{3} + i$$

<u>Example</u>: Evaluate $\int_0^\pi e^{it}dt$.

Solution:

$$\int_0^\pi e^{it}dt = \int_0^\pi (\cos t + i\sin t)dt = \int_0^\pi \cos t\,dt + i\int_0^\pi \sin t\,dt$$
$$= \sin t\,|_0^\pi - i\cos t\,|_0^\pi = 0 - i(-1-1) = 2i$$

<u>Theorem</u>: Let $w(t) = u(t) + iv(t)$ and $W(t) = U(t) + iV(t)$ such that $W'(t) = w(t)$, then $\int_a^b w(t)dt = W(t)|_a^b = W(b) - W(a)$

Proof: $W'(t) = w(t) \;\Rightarrow\; U'(t) = u(t),\; V'(t) = v(t)$

$$\int_a^b w(t)dt = \int_a^b u(t)dt + i\int_a^b v(t)dt = U(t)|_a^b + iV(t)|_a^b$$
$$= \big(U(t) + i\,V(t)\big)|_a^b = W(t)|_a^b = W(b) - W(a).$$

<u>Example</u>: Evaluate $\int_0^1 (1+it)^2 dt$.

Solution:

$$\int_0^1 (1+it)^2 dt = \frac{1}{i}\int_0^1 (1+it)^2\, i\,dt = \frac{1}{i}\frac{(1+it)^3}{3}|_0^1 = \frac{1}{i}\left[\frac{(1+i)^3}{3} - \frac{(1)^3}{3}\right] = \frac{2}{3} + i$$

<u>Example</u>: Evaluate $\int_0^\pi e^{it}dt$.

Solution:

$$\int_0^\pi e^{it}dt = \frac{1}{i}\int_0^\pi e^{it}\, i\,dt = \frac{e^{it}}{i}|_0^\pi = \frac{e^{i\pi}}{i} - \frac{1}{i} = \frac{1}{i}[(\cos\pi + i\sin\pi) - 1] = 2i.$$

Proposition: Let $f(t) = u_1(t) + iv_1(t)$ and $g(t) = u_2(t) + iv_2(t)$ are complex- valued functions of a real variable t continuous on an interval $a \leq t \leq b$, then

i. $\int_a^b kf(t)dt = k\int_a^b f(t)dt,\;\; k$ a complex constant,

ii. $\int_a^b (f(t) + g(t))dt = \int_a^b f(t)dt + \int_a^b g(t)dt$,

iii. $\int_a^b f(t)dt = \int_a^c f(t)dt + \int_c^b f(t)dt, \quad c \in [a, b]$

iv. $\int_a^b f(t)dt = -\int_b^a f(t)dt$

Exercises:

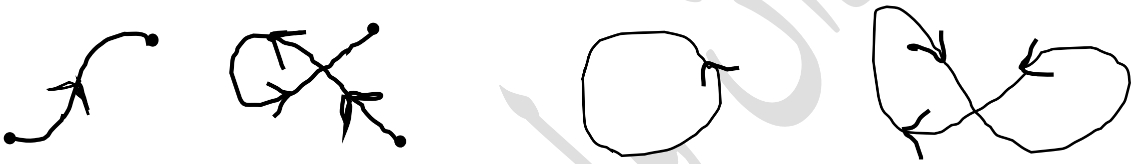
1) Evaluate $\int_0^{\frac{\pi}{4}} e^{it}dt.$
2) Evaluate $\int_1^2 (\frac{1}{t} - i)^2 dt.$
3) Evaluate $\int_0^{\frac{\pi}{6}} e^{i2t}dt.$
4) Evaluate $\int_1^2 (\frac{1}{t-i})^2 dt.$
5) Evaluate $\int_0^{\infty} e^{-zt}dt.$
6) Evaluate $Re\int_0^{\pi} e^{(1+i)t}dt$ and $Im\int_0^{\pi} e^{(1+i)t}dt.$

2. Contour Integral.

Definition: If $z = z(t) = x(t) + iy(t), a \leq t \leq\ b$, if $x(t), y(t)$, are continuous functions on a closed interval $[a, b]$. Then $z(t)$ is called path and denoted it by C. $z(a)$ is called initial point and $z(b)$ is terminal point.

Remark:

1. $'C'$ is a closed path if $z(a) = z(b)$.
2. $'C'$ is a simple path if $z(t_1) = z(t_2) \Rightarrow t_1 = t_2$ or $t_1 \neq t_2 \Rightarrow z(t_1) \neq z(t_2)$.
3. If $z(t_1) \neq z(t_2)$ when $t_1 \neq t_2$, also $z(a) = z(b)$ then the path C is called simple closed path.
4. $'C'$ is a smooth path if and only if
   i) Both x and y are continuously differentiable
   ii) For all $t \in [a, b]$ either $x'(t) \neq 0$ or $y'(t) \neq 0$.

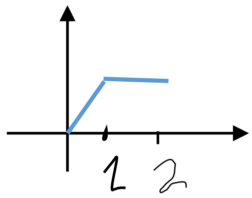


Simple Not closed — *Not Simple Not closed* — *Simple Closed* — *Not Simple Closed*

Example: $C: z(t) = \begin{cases} t + it & 0 \leq t \leq 1 \\ t + i & 1 \leq t \leq 2 \end{cases}$, is a simple path but not closed.

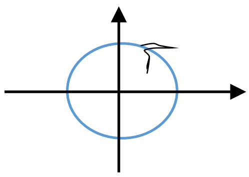
$0 \leq t \leq 1: \quad x = t, y = t \Rightarrow x = y$

$1 \leq t \leq 2: x = t, y = 1$



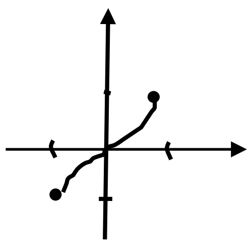
Example: $C: z(\theta) = e^{i\theta} \quad 0 \leq \theta \leq 2\pi$, is a simple closed path.

$e^{i\theta} = \cos\theta + i\sin\theta$
$z(0) = 1,\ z(2\pi) = 1$ unit circle



Example: $C: z(t) = t + it^3 \quad -1 \leq t \leq 1$, is a simple path but not closed.

$x = t, y = t^3 \Rightarrow y = x^3$
$z(-1) = -1 - i$
$z(1) = 1 + i$



Definition: A **contour** is a piecewise smooth path C, (i.e.) a curve consisting of a finite number of smooth paths joined end to end.

Definition: If f is continuous on a smooth curve C given by the parameterization , $z(t) = x(t) + iy(t) \quad a \le t \le b$, then

$$\int_C f(z)dz = \int_a^b f\big(z(t)\big)z'(t)dt.$$

Properties:

1. $\int_C z_0 f(z)dz = z_0 \int_C f(z)dz$, z_0 is a complex constant,
2. $\int_{-C} f(z)dz = -\int_C f(z)dz$
3. $\int_{C_1+C_2} f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz$
4. $\int_C (f(z) + g(z))dz = \int_C f(z)dz + \int_C g(z)dz$,
5. If $C = C_1 + (-C_2)$

$$\int_C f(z)dz = \int_{(C_1+(-C_2))} f(z)dz = \int_{C_1} f(z)dz + \int_{-C_2} f(z)dz$$
$$= \int_{C_1} f(z)dz - \int_{C_2} f(z)dz$$

Example : Evaluate $\int_C \bar{z}dz$, where C is given by $x = 3t, y = t^2, -1 \le t \le 4$.

Solution: $z(t) = 3t + it^2 \quad \Rightarrow \quad z'(t) = 3 + 2it$

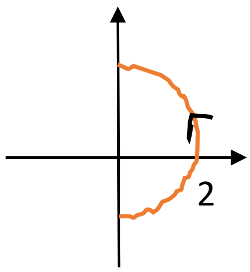
$f(z) = \bar{z}$

We have $f(z(t)) = \overline{3t + t^2} = 3t - it^2$

Therefor the integral is:

$$\int_C \bar{z}dz = \int_{-1}^{4}(3t - it^2)(3 + 2it)\,dt = \int_{-1}^{4}(2t^3 + 9t + 3it^2)\,dt$$
$$= \int_{-1}^{4}(2t^3 + 9t)\,dt + i\int_{-1}^{4}(3t^2)\,dt$$
$$= \left(\frac{1}{2}t^4 + \frac{9}{2}t^2\right)\Big|_{-1}^{4} + it^3\Big|_{-1}^{4} = 195 + 65i.$$

Example : Evaluate $\int_C \bar{z}dz$, where C is the right hand half of the circle $|z| = 2$ from $z = -2i$ to $z = 2i$.

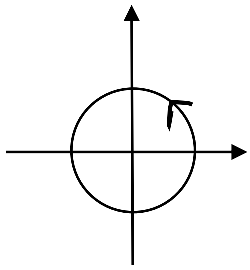


Solution: $z = 2e^{i\theta} \Rightarrow \bar{z} = 2e^{-i\theta} \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$dz = 2ie^{i\theta}d\theta$

$$\int_C f(z)dz = \int_C f\big(z(t)\big)z'(t)dt$$
$$= \int_C f\big(z(\theta)\big)z'(\theta)d\theta$$
$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2\,e^{-i\theta}.i2e^{i\theta}d\theta = 4i\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta = 4i\left(\frac{\pi}{2} + \frac{\pi}{2}\right) = 4i\pi.$$

Example : Evaluate $\oint_C \frac{1}{z}dz$, where C is the circle $x = \cos t,\ y = \sin t$ $0 \le t \le 2\pi$.



Solution:

In this case $z(t) = \cos t + i\sin t = e^{it}$, and $f\big(z(t)\big) = \frac{1}{z(t)} = e^{-it}$.

Hence,

$$\oint_C \frac{1}{z}dz = \int_0^{2\pi}\big(e^{-it}\big)ie^{it}dt = i\int_0^{2\pi} dt = 2\pi i.$$

Example : Evaluate $\int_C (x^2 + iy^2)dz$, where $C: z(t) = \begin{cases} t + it & 0 \le t \le 1 \\ 1 + it & 1 \le t \le 2 \end{cases}$

Solution:

$$\int_C (x^2 + iy^2)dz = \int_{C_1}(x^2 + iy^2)dz + \int_{C_2}(x^2 + iy^2)dz$$

$$\int_{C_1}(x^2 + iy^2)dz = \int_0^1 (t^2 + it^2)(1 + i)dt$$

$$= (1 + i)^2\int_0^1 t^2 dt = \frac{(1+i)^2}{3} = \frac{2}{3}i$$

$$\int_{C_2}(x^2 + iy^2)dz = \int_1^2 (1 + it^2)idt$$

$$= -\int_1^2 t^2 dt + i\int_1^2 dt = -\frac{7}{3} + i$$

Therefore,

$$\int_C (x^2 + iy^2)dz = \frac{2}{3}i + \left(-\frac{7}{3} + i\right) = -\frac{7}{3} + \frac{5}{3}i.$$

Exercises: Evaluate the given integral along the indicated contour.

1) $\int_C (z + 3)dz$, where C is $x = 2t, y = 4t - 1, 1 \le t \le 3$
2) $\int_C z^2 dz$, where C is $z(t) = 3t + 2it, -2 \le t \le 2$
3) $\int_C \frac{z+1}{z}dz$, where C is the right hand half of the circle $|z| = 1$ from $z = -i$ to $z = i$
4) $\oint_C z^2 dz$, where C is the path formed by the line segment from -3 to 3 and the semicircle located above the line segment.